Duality formulas for robust pricing and hedging in discrete time*

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Abstract

In this paper we derive robust super- and subhedging dualities for contingent claims that can depend on several underlying assets. In addition to strict super- and subhedging, we also consider relaxed versions which, instead of eliminating the shortfall risk completely, aim to reduce it to an acceptable level. This yields robust price bounds with tighter spreads. As applications we study strict super- and subhedging with general convex transaction costs and trading constraints as well as risk based hedging with respect to robust versions of the average value at risk and entropic risk measure. Our approach is based on representation results for increasing convex functionals and allows for general financial market structures. As a side result it yields a robust version of the fundamental theorem of asset pricing.

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1. Introduction

Super- and subhedging dualities lie at the heart of no-arbitrage arguments in quantitative finance. By relating prices to hedging, they provide bounds on arbitrage-free prices. But they also serve as a stepping stone to the application of duality methods to portfolio optimization problems. In traditional financial modeling, uncertainty is described by a single probability measure $\mathbb{P}$, and the super- and subhedging prices of a contingent claim X are given by

$$\phi(X) = \inf \{m \in \mathbb{R} : \text{there exists a } Y \in G \text{ such that } m - X + Y \geq 0 \ \mathbb{P}\text{-a.s.}\} \tag{1.1}$$

and

$$-\phi(-X) = \sup \{m \in \mathbb{R} : \text{there exists a } Y \in G \text{ such that } X - m + Y \geq 0 \ \mathbb{P}\text{-a.s.}\}, \tag{1.2}$$

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where G is the set of all realizable trading gains. Classical results assume that X depends on a set of underlying assets which can be traded dynamically without transaction costs or constraints. Then G is a linear space, and under a suitable no-arbitrage condition, one obtains dualities of the form

$$\phi(X) = \sup_{\mathbb{Q} \in \mathcal{M}^e(\mathbb{P})} \mathbb{E}^{\mathbb{Q}} X \quad \text{and} \quad -\phi(-X) = \inf_{\mathbb{Q} \in \mathcal{M}^e(\mathbb{P})} \mathbb{E}^{\mathbb{Q}} X, \tag{1.3}$$

where $\mathcal{M}^e(\mathbb{P})$ is the set of all (local) martingale measures equivalent to $\mathbb{P}$, see e.g. [27] or [19] for an overview.

In this paper we do not assume that the probabilities of all future events are known. So instead of starting with a predefined probability measure, we specify a collection of possible trajectories for a set of basic assets. This includes a wide range of setups, from a single binomial tree model to the model-free case, in which at any time, the prices of all assets can lie anywhere in $\mathbb{R}_+$ (or $\mathbb{R}$). We replace the $\mathbb{P}$-almost sure inequalities in (1.1) and (1.2) by a general set of acceptable positions A and consider super- and subhedging functionals of the form

$$\phi(X) = \inf \{m \in \mathbb{R} : m - X \in A - G\} \quad \text{and} \quad -\phi(-X) = \sup \{m \in \mathbb{R} : X - m \in A - G\}. \tag{1.4}$$

If A is the cone of non-negative positions, this describes strict super- and subhedging, which requires that the shortfall risk be eliminated completely. Alternatively, one can allow for a certain amount of risk by enlarging the set A. This reduces the spread between super- and subhedging prices. Moreover, the set of trading gains G does not have to be a linear space and can describe general market structures with transaction costs and trading constraints.

Our main result, Theorem 2.1, yields dual expressions for the quantities in (1.4) in terms of expected values under the assumption that it is not possible to transform a debt into an acceptable position by investing in the market. As a byproduct, it contains a robust fundamental theorem of asset pricing (FTAP), which relates two different notions of no-arbitrage to the existence of generalized martingale measures. In the case where the underlying assets are bounded, it holds for general sets A and G such that $G - A$ is convex. Otherwise, it needs that the set $G - A$ is large enough. This can be guaranteed by either assuming that the financial market is sufficiently rich or, as shown in Proposition 2.2, that the acceptability condition is not too strict. In Sections 3 and 4 we study different specifications of A and G, for which the dual representations can be computed explicitly. Section 3 is devoted to the case where A consists of all non-negative positions, corresponding to strict super- and subhedging. We consider general semi-static trading strategies consisting of dynamic investments in the underlying assets and static derivative positions. Proposition 3.1 covers general convex transaction costs and constraints on the derivative positions. Proposition 3.2 deals with dynamic shortselling constraints. In Section 4 we relax the hedging requirement and control shortfall risk with a family of risk measures defined in different probabilistic models. This allows to introduce risk-tolerance in a setup of model-uncertainty and makes it possible to specify attitudes towards risk and ambiguity. Our price bounds then become robust good deal bounds. Proposition 4.2 gives an explicit duality formula in the case where risk is assessed with a robust average value at risk. Proposition 4.4 provides the same for a robust entropic risk measure.

Our approach is based on representation results for increasing convex functionals and permits to combine robust methods with transaction costs, trading constraints, partial hedging and good deal bounds. Robust hedging methods go back to [31] and were further investigated in e.g. [16, 18, 37]. Various versions of robust FTAPs and superhedging dualities have been derived in [1, 2, 3, 4, 7, 8, 9, 12, 13, 21, 22, 23, 28, 38]. For FTAPs and superhedging dualities under transaction costs we refer to [20, 30, 33, 40] and the references therein. The literature on partial hedging started with the quantile-hedging approach of [25] and subsequently developed more general risk-based methods; see e.g. [10, 26, 39] and the closely related literature on good deals, such as e.g. [6, 32, 35, 41].

The rest of the paper is organized as follows: Section 2 introduces the notation and states the paper's main results. In Section 3 we study strict super- and subhedging and give robust FTAPs with corresponding superhedging dualities under convex transaction costs and trading constraints. As special cases we obtain versions of Kantorovich's transport duality [34] that include martingale or supermartingale constraints. In Section 4 we use robust risk measures to weaken the acceptability condition. This yields robust good deal bounds lying closer together than the strict super- and subhedging prices of Section 3. All proofs are given in the appendix.

2. Main results

We consider a model with finitely many trading periods $t = 0, 1, \dots, T$ and $J + 1$ financial assets $S^0, \dots, S^J$. As sample space we take a non-empty closed subset Ω of $\mathbb{R}^{JT}$. S^0 is used as numéraire, and the prices of the assets S^j, $j \geq 1$, in units of S^0 are modeled as

$$S^j_0 = s^j_0 \in \mathbb{R} \quad \text{and} \quad S^j_t(\omega) = \omega^j_t, \quad t \geq 1,\ \omega \in \Omega.$$

We endow Ω with the Euclidean metric and denote the space of all Borel measurable functions $X : \Omega \to \mathbb{R}$ by B. We assume that the set of realizable trading gains is given by a subset $G \subseteq B$ containing 0 and the set of acceptable positions by a subset $A \subseteq B$ containing the positive cone $B^+ := \{X \in B : X \geq 0\}$ such that $G - A$ is convex. The corresponding superhedging functional is

$$\phi(X) := \inf \{m \in \mathbb{R} : m - X \in A - G\}.$$

It determines for every liability $X \in B$, the smallest amount of money m needed such that $m - X$ can be transformed into an acceptable position by investing in the financial market. The case $A = B^+$ corresponds to strict superhedging, which requires that a contingent claim be superreplicated in every possible scenario. A larger acceptance set A relaxes the superhedging requirement and lowers the hedging costs. The subhedging functional induced by G and A is given by

$$-\phi(-X) = \sup \{m \in \mathbb{R} : X - m \in A - G\}.$$

In the whole paper, $Z : \Omega \to [1, +\infty)$ is a continuous function with bounded sublevel sets $\{\omega \in \Omega : Z(\omega) \leq z\}$, $z \in \mathbb{R}_+$. Let $B_Z \subseteq B$ be the subspace of positions $X \in B$ such that X/Z is bounded, U_Z the set of all upper semicontinuous $X \in B_Z$ and C_Z the space of all continuous $X \in B_Z$. By $\mathcal{P}_Z$ we denote the set of all Borel probability measures on Ω satisfying the integrability condition $\mathbb{E}^{\mathbb{P}} Z < +\infty$. Define the convex conjugate $\phi^* : \mathcal{P}_Z \to \mathbb{R} \cup \{\pm\infty\}$ by

$$\phi^*(\mathbb{P}) := \sup_{X \in C_Z} (\mathbb{E}^{\mathbb{P}} X - \phi(X)).$$

Then the following holds:

Theorem 2.1. *Assume*

$$\textit{for every } n \in \mathbb{N}, \textit{ there exists a } z \in \mathbb{R}_+ \textit{ such that } n(Z - z)^+ - \frac{1}{n} \in G - A. \tag{2.1}$$

Then the following three conditions are equivalent:

(i) $\mathbb{R}_+ \cap (G - A) = \{0\}$

(ii) *there exists a probability measure* $\mathbb{P} \in \mathcal{P}_Z$ *such that* $\mathbb{E}^{\mathbb{P}} X \leq 0$ *for all* $X \in C_Z \cap (G - A)$

(iii) ϕ *is real-valued on* B_Z *with* $\phi(0) = 0$ *and* $\phi(X) = \max_{\mathbb{P} \in \mathcal{P}_Z} (\mathbb{E}^{\mathbb{P}} X - \phi^*(\mathbb{P}))$ *for all* $X \in C_Z$.

If in addition to (2.1)*, one has*

$$\phi(X) = \inf_{Y \in C_Z, Y \geq X} \phi(Y) \quad \textit{for all } X \in U_Z, \tag{2.2}$$

then (i)–(iii) *are also equivalent to each of the following three:*

(iv) *there exists no* $X \in G - A$ *such that* $X(\omega) > 0$ *for all* $\omega \in \Omega$

(v) *there exists a probability measure* $\mathbb{P} \in \mathcal{P}_Z$ *such that* $\mathbb{E}^{\mathbb{P}} X \leq 0$ *for all* $X \in U_Z \cap (G - A)$

(vi) ϕ *is real-valued on* B_Z *with* $\phi(0) = 0$ *and* $\phi(X) = \max_{\mathbb{P} \in \mathcal{P}_Z} (\mathbb{E}^{\mathbb{P}} X - \phi^*(\mathbb{P}))$ *for all* $X \in U_Z$.

(iii) and (vi) yield dual representations for the superhedging functional ϕ. They directly translate into dual representations for the subhedging functional $-\phi(-X)$. If condition (iii) holds, one has

$$-\phi(-X) = \min_{\mathbb{P} \in \mathcal{P}_Z} (\mathbb{E}^{\mathbb{P}} X + \phi^*(\mathbb{P})) \quad \text{for all } X \in C_Z,$$

and the representation extends to all $X \in U_Z$ if (vi) is satisfied. Moreover, note that as soon as ϕ is real-valued on B_Z with $\phi(0) = 0$, the same is true for the subhedging functional, and one obtains by convexity, $\phi(X) + \phi(-X) \geq 2\phi(0) = 0$, yielding the ordering

$$\phi(X) \geq -\phi(-X) \quad \text{for all } X \in B_Z.$$

(i) and (iv) are no-arbitrage conditions, or in the case where the acceptance set A is larger than B^+, no-good deal conditions. (i) means that there exists no trading strategy starting with zero initial capital generating an outcome that exceeds an acceptable position by a positive constant, or equivalently, that it is not possible to turn a debt into an acceptable position by investing in the market. The same condition was used by [29] and [36] in the standard setup. (iv) is stronger. In the case where A equals B^+, it corresponds to absence of model-independent arbitrage as introduced by [17] and used e.g., in [1].

(ii) and (v) generalize the concept of a martingale measure. For instance, if $A = B^+$ and the underlying assets are liquidly traded, they consist of proper martingale measures. But in the presence of proportional transaction costs, they become ε-approximate martingale measures, and under short-selling constraints, supermartingale measures (see the examples in Section 3).

A wide class of acceptance sets can be written as

$$A = \left\{ X \in B_Z : \mathbb{E}^{\mathbb{P}} X + \alpha(\mathbb{P}) \geq 0 \text{ for all } \mathbb{P} \in \mathcal{P}_Z \right\} + B^+ \tag{2.3}$$

for a suitable mapping $\alpha : \mathcal{P}_Z \to \mathbb{R}_+ \cup \{+\infty\}$. In the extreme case $\alpha \equiv 0$, A is the positive cone B^+. On the other hand, it can be shown that if α grows fast enough, assumption (2.1) of Theorem 2.1 is automatically satisfied:

Proposition 2.2. *Condition* (2.1) *holds if* A *is given by* (2.3) *for a mapping* $\alpha : \mathcal{P}_Z \to \mathbb{R}_+ \cup \{+\infty\}$ *satisfying*

(A1) $\inf_{\mathbb{P}\in\mathcal{P}_Z} \alpha(\mathbb{P}) = 0$ *and*

(A2) $\alpha(\mathbb{P}) \geq \mathbb{E}^{\mathbb{P}}\beta(Z)$ *for all* $\mathbb{P} \in \mathcal{P}_Z$*, where* $\beta : [1, +\infty) \to \mathbb{R}$ *is an increasing*[1] *function with the property* $\lim_{x\to+\infty} \beta(x)/x = +\infty$.

The following result gives a dual condition for assumption (2.2) which will be useful in Sections 3 and 4.

Proposition 2.3. *Assume* (2.1) *holds. Then*

$$\phi^*(\mathbb{P}) = \sup_{X\in C_Z\cap(G-A)} \mathbb{E}^{\mathbb{P}}X \leq \sup_{X\in U_Z} (\mathbb{E}^{\mathbb{P}}X - \phi(X)) = \sup_{X\in U_Z\cap(G-A)} \mathbb{E}^{\mathbb{P}}X \quad \text{for all } \mathbb{P} \in \mathcal{P}_Z,$$

and the inequality is an equality if and only if ϕ *satisfies* (2.2).

3. Strict superhedging

In this section we concentrate on the case where the acceptance set A is the positive cone B^+. Ω is assumed to be a non-empty closed subset of $\mathbb{R}^{JT}_+$, and the price processes $S^1, \dots, S^J$ are given by $S^j_0 = s^j_0 \in \mathbb{R}$ and $S^j_t(\omega) = \omega^j_t$, $t \geq 1$, $\omega \in \Omega$. They generate the filtration $\mathcal{F}_t = \sigma(S^j_s : j = 1, \dots, J,\ s \leq t)$, $t = 0, \dots, T$. As growth function we choose $Z = 1 + \sum_{j,t} S^j_t$. Then B_Z is the space of all measurable functions $X : \Omega \to \mathbb{R}$ of linear growth, and for any set $G \subseteq B$ of trading gains containing 0 such that $G - B^+$ is convex, condition (2.1) is equivalent to

$$\begin{array}{l} \text{for every } n \in \mathbb{N},\ j = 1, \dots, J, \text{ and } t = 1, \dots, T, \text{ there exist} \\ X \in G \text{ and } K \in \mathbb{R}_+ \text{ such that } X \geq n(S^j_t - K)^+ - 1/n. \end{array} \tag{3.1}$$

Obviously, (3.1) holds automatically if Ω is compact.

Let us define

$$\phi^*_G(\mathbb{P}) := \sup_{X\in G} \mathbb{E}^{\mathbb{P}}X, \quad \mathbb{P} \in \mathcal{P}_Z,$$

with the understanding that

$$\mathbb{E}^{\mathbb{P}}X := \begin{cases} \lim_{m\to+\infty} \mathbb{E}^{\mathbb{P}}(X \wedge m) & \text{if } \mathbb{E}^{\mathbb{P}}X^- < +\infty \\ -\infty & \text{otherwise,} \end{cases}$$

and introduce the corresponding set of generalized martingale measures

$$\mathcal{M} := \{\mathbb{P} \in \mathcal{P}_Z : \phi^*_G(\mathbb{P}) = 0\} = \left\{\mathbb{P} \in \mathcal{P}_Z : \mathbb{E}^{\mathbb{P}}X \leq 0 \text{ for all } X \in G\right\}.$$

3.1. Semi-static hedging with convex transaction costs and constraints

Let us first assume that the assets $S^1, \dots, S^J$ can be traded dynamically according to any J-dimensional predictable strategy $(\vartheta_t)^T_{t=1}$, and in addition, it is possible to form a static portfolio of derivatives with payoffs $H_i \in C_Z$ for i in an index set I. We denote by $\mathbb{R}^I_0$ the set of vectors in $\mathbb{R}^I$ with at most finitely many components different from 0 and suppose that the static part of the strategy θ is constrained to lie in a given convex subset $\Theta \subseteq \mathbb{R}^I_0$ containing 0. We model

[1] We call a function f from a subset $I \subseteq \mathbb{R}$ to $\mathbb{R}$ increasing if $f(x) \geq f(y)$ for $x \geq y$.

transactions costs in the derivatives market with a convex mapping $h : \Theta \to C_Z$ satisfying $h(0) = 0$. Transaction costs arising from dynamic trading are, as in [2, 11, 20], given by continuous functions $g_t^j : \Omega \times \mathbb{R} \to \mathbb{R}$, $j = 1, \ldots, J$, $t = 0, \ldots, T-1$, such that $g_t^j(\omega, x)$ is $\mathcal{F}_t$-measurable in ω and convex in x with $g_t^j(\omega, 0) = 0$. In the case where $\Omega \subseteq \mathbb{R}^{JT}$ is unbounded, we also assume that $\sup_{x \in E} |g_t^j(\omega, x)|/Z(\omega)$ is bounded in ω for every bounded subset $E \subseteq \mathbb{R}$. As usual, we suppress the ω-dependence of g_t^j in the notation. Then the resulting set of trading gains G consists of outcomes of the form

$$\sum_{t=1}^{T} \sum_{j=1}^{J} \left(\vartheta_t^j \Delta S_t^j - g_{t-1}^j(\Delta \vartheta_t^j S_{t-1}^j) \right) + \sum_{i \in I} \theta_i H_i - h(\theta),$$

where $\Delta S_t^j = S_t^j - S_{t-1}^j$ and $\Delta \vartheta_t^j = \vartheta_t^j - \vartheta_{t-1}^j$ with $\vartheta_0^j = 0$.

Under these circumstances we obtain the following version of Theorem 2.1:

Proposition 3.1. *If* (3.1) *holds, the following four conditions are equivalent:*

(i) *there exist no $X \in G$ and $\varepsilon \in \mathbb{R}_+ \setminus \{0\}$ such that $X \geq \varepsilon$*

(ii) *there exists no $X \in G$ such that $X(\omega) > 0$ for all $\omega \in \Omega$*

(iii) $\mathcal{M} \neq \emptyset$

(iv) *ϕ is real-valued on B_Z with $\phi(0) = 0$ and $\phi(X) = \max_{\mathbb{P} \in \mathcal{P}_Z} (\mathbb{E}^{\mathbb{P}} X - \phi_G^*(\mathbb{P}))$ for all $X \in U_Z$.*

Moreover,

$$\phi_G^*(\mathbb{P}) = \begin{cases} \sum_{t=0}^{T-1} \sum_{j=1}^{J} \int_{\{S_t^j > 0\}} g_t^{j*} \left(\frac{\mathbb{E}^{\mathbb{P}}[S_T^j | \mathcal{F}_t] - S_t^j}{S_t^j} \right) d\mathbb{P} + h^*(\mathbb{P}) & \text{if } \mathbb{P} \in \mathcal{R} \\ +\infty & \text{if } \mathbb{P} \notin \mathcal{R}, \end{cases} \tag{3.2}$$

for

$$g_t^{j*}(y) := \sup_{x \in \mathbb{R}} (xy - g_t^j(x)), \quad h^*(\mathbb{P}) := \sup_{\theta \in \Theta} \mathbb{E}^{\mathbb{P}} \left(\sum_{i \in I} \theta_i H_i - h(\theta) \right),$$

and

$$\mathcal{R} := \left\{ \mathbb{P} \in \mathcal{P}_Z : \mathbb{P}[S_t^j = 0 \text{ and } S_T^j > 0] = 0 \text{ for all } j = 1, \ldots, J \text{ and } t \leq T-1 \right\}.$$

Proposition 3.1 extends results of [1, 2, 22]. [1] and [22] consider proportional transaction costs and make an assumption similar to condition (3.1). [2] treat the case of general convex transaction costs in a setup with a bounded sample space Ω.

3.1.1. Proportional transaction costs

As a special case, let us consider proportional transaction costs and no trading constraints; that is, $\Theta = \mathbb{R}_0^I$ and $h(\theta) = \sum_{i \in I} h_i^+ \theta_i^+ - h_i^- \theta_i^-$ for bid and ask prices $h_i^-, h_i^+ \in \mathbb{R}$. Moreover, g_t^j is of the form $g_t^j(x) = \varepsilon_t^j |x|$ for $\mathcal{F}_t$-measurable random coefficients $\varepsilon_t^j \in C_Z^+$. The corresponding trading outcomes are of the form

$$\sum_{t=1}^{T} \sum_{j=1}^{J} \left(\vartheta_t^j \Delta S_t^j - \varepsilon_{t-1}^j |\Delta \vartheta_t^j S_{t-1}^j| \right) + \sum_{i \in I} \left(\theta_i H_i - \theta_i^+ h_i^+ + \theta_i^- h_i^- \right),$$

and

$$g_t^{j*}(y) = \begin{cases} 0 & \text{if } |y| \le \varepsilon_t^j \\ +\infty & \text{otherwise,} \end{cases} \qquad h^*(\mathbb{P}) = \begin{cases} 0 & \text{if } h_i^- \le \mathbb{E}^{\mathbb{P}} H_i \le h_i^+ \text{ for all } i \in I \\ +\infty & \text{otherwise.} \end{cases}$$

By Proposition 3.1, one has

$$\phi_G^*(\mathbb{P}) = \begin{cases} 0 & \text{if } \mathbb{P} \in \mathcal{M} \\ +\infty & \text{otherwise,} \end{cases}$$

where $\mathcal{M}$ is the set of all probability measures $\mathbb{P} \in \mathcal{P}_Z$ satisfying the conditions

a) $(1 - \varepsilon_t^j) S_t^j \le \mathbb{E}^{\mathbb{P}}[S_T^j \mid \mathcal{F}_t] \le (1 + \varepsilon_t^j) S_t^j$ for all $j = 1, \dots, J$ and $t = 0, \dots, T-1$

b) $h_i^- \le \mathbb{E}^{\mathbb{P}} H_i \le h_i^+$ for all $i \in I$.

So if (3.1) and (i) of Proposition 3.1 hold, one obtains the duality

$$\phi(X) = \max_{\mathbb{P} \in \mathcal{M}} \mathbb{E}^{\mathbb{P}} X \quad \text{for all } X \in U_Z. \tag{3.3}$$

If $\varepsilon_t^j \equiv 0$, dynamic trading is frictionless, and a) reduces to the standard martingale condition. In this case, (3.3) is the superhedging duality derived in [1]. On the other hand, in the case where the coefficients ε_t^j are constant and $(H_i)_{i \in I}$ consists of European options depending continuously on S_T, (3.3) becomes the superhedging duality of [22].

3.1.2. Superlinear transaction costs

For $\Theta = \mathbb{R}_0^I$ and transaction costs corresponding to

$$g_t^j(x) = \frac{\varepsilon_t^j}{p_j} |x|^{p_j}, \quad h(\theta) = \sum_{i \in I} h_i \theta_i + \frac{\delta_i}{q_i} |\theta_i|^{q_i}$$

for positive $\mathcal{F}_t$-measurable $\varepsilon_t^j \in C_Z$ and constants $\delta_i > 0$, $p_j, q_j > 1$, $h_i \in \mathbb{R}$, one obtains from Proposition 3.1,

$$\phi_G^*(\mathbb{P}) = \sum_{t=1}^{T} \sum_{j=1}^{J} \frac{(\varepsilon_{t-1}^j)^{1-p_j'}}{p_j'} \mathbb{E}^{\mathbb{P}} \left| \frac{\mathbb{E}^{\mathbb{P}}[S_T^j \mid \mathcal{F}_{t-1}] - S_{t-1}^j}{S_{t-1}^j} \right|^{p_j'} + \sum_{i \in I} \frac{\delta_i^{1-q_i'}}{q_i'} \left| \mathbb{E}^{\mathbb{P}} H_i - h_i \right|^{q_i'},$$

where $p_j' := p_j/(p_j - 1)$, $q_i' := q_i/(q_i - 1)$, $0/0 := 0$ and $x/0 := +\infty$ for $x > 0$. Moreover, if (3.1) and condition (i) of Proposition 3.1 hold, one has

$$\phi(X) = \max_{\mathbb{P} \in \mathcal{P}_Z} (\mathbb{E}^{\mathbb{P}} X - \phi_G^*(\mathbb{P})) \quad \text{for all } X \in U_Z.$$

3.1.3. European call options and constraints on the marginal distributions

Let $g_t^j : \Omega \times \mathbb{R} \to \mathbb{R}$ be as in the beginning of Subsection 3.1 above and assume the family $(H_i)_{i \in I}$ consists of all European call options

$$(S_t^j - K)^+, \quad j = 1, \dots J, \ t = 1, \dots, T, \ K \in \mathbb{R}_+.$$

Moreover suppose that arbitrary quantities of $(S^j_t - K)^+$ can be sold or bought at prices $p^{j,-}_{t,K}$ and $p^{j,+}_{t,K}$, respectively. If $\lim_{K\to+\infty} p^{j,+}_{t,K} = 0$ for all j and t, condition (3.1) holds. So if in addition, (i) of Proposition 3.1 is satisfied, one obtains

$$\phi(X) = \max_{\mathbb{P}} \left(\mathbb{E}^{\mathbb{P}} X - \sum_{t=0}^{T-1} \sum_{j=1}^{J} \int_{\{S^j_t > 0\}} g^{j*}_t \left(\frac{\mathbb{E}^{\mathbb{P}}[S^j_T \mid \mathcal{F}_t] - S^j_t}{S^j_t} \right) d\mathbb{P} \right) \quad \text{for all } X \in U_Z, \tag{3.4}$$

where the maximum is over all $\mathbb{P} \in \mathcal{P}_Z$ such that

a) $\mathbb{P}[S^j_t \text{ and } S^j_T > 0] = 0$ for all $j = 1, \dots, J$ and $t = 0, \dots, T-1$

b) $p^{j,-}_{t,K} \le \mathbb{E}^{\mathbb{P}}(S^j_t - K)^+ \le p^{j,+}_{t,K}$ for all $j = 1, \dots, J$, $t = 1, \dots T$ and $K \in \mathbb{R}_+$.

$\mathbb{E}^{\mathbb{P}}(S^j_t - K)^+$ can be written as $\int_K^{+\infty} \mathbb{P}[S^j_t > x]dx$. So condition b) puts constraints on the distributions of S^j_t under $\mathbb{P}$, and in the limiting case $p^{j,-}_{t,K} = p^{j,+}_{t,K}$, it fully determines the distributions of S^j_t under $\mathbb{P}$. In particular, if $g^j_t \equiv 0$ and $p^{j,-}_{t,K} = p^{j,+}_{t,K} = p^j_{t,K} \in \mathbb{R}_+$ for all j, t, K, it follows from (3.1) and (i) of Proposition 3.1 that there exist unique marginal distributions ν^j_t on $\mathbb{R}_+$ specified by $\int_K^{+\infty} \nu^j_t(x, +\infty)dx = p^j_{t,K}$, $K \in \mathbb{R}_+$, such that

$$\phi(X) = \max_{\mathbb{P} \in \mathcal{M}} \mathbb{E}^{\mathbb{P}} X \quad \text{for all } X \in U_Z, \tag{3.5}$$

where $\mathcal{M}$ consists of all $\mathbb{P} \in \mathcal{P}_Z$ satisfying

a) $S^1, \dots, S^J$ are martingales under $\mathbb{P}$

b) the distribution of S^j_t under $\mathbb{P}$ is ν^j_t for all $j = 1, \dots, J$ and $t = 1, \dots, T$.

(3.5) is a variant of Kantorovich's transport duality [34] and has recently been studied in different setups under the name martingale transport duality; see e.g., [3, 4, 28].

3.2. Semi-static hedging with short-selling constraints

Now assume that dynamic trading does not incur transaction costs, but only non-negative quantities of the assets $S^1, \dots, S^J$ can be held. As above, one can invest statically in derivatives with payoffs $H_i \in C_Z$, $i \in I$, according to a strategy θ lying in a convex subset $\Theta \subseteq \mathbb{R}^I_0$ containing 0. Let $h : \Theta \to C_Z$ be a convex mapping with $h(0) = 0$ and suppose that G consists of outcomes of the form

$$\sum_{t=1}^{T} \sum_{j=1}^{J} \vartheta^j_t \Delta S^j_t + \sum_{i \in I} \theta_i H_i - h(\theta),$$

where $(\vartheta_t)_{t=1}^T$ is a non-negative J-dimensional predictable strategy and $\theta \in \Theta$. Then the following variant of Proposition 3.1 holds:

Proposition 3.2. *If* (3.1) *is satisfied, the conditions* (i)–(iv) *of Proposition 3.1 are equivalent, where*

$$\phi^*_G(\mathbb{P}) = \begin{cases} \sup_{\theta \in \Theta} \mathbb{E}^{\mathbb{P}}\left(\sum_{i \in I} \theta_i H_i - h(\theta)\right) & \text{if } S^1, \dots, S^J \text{ are supermartingales under } \mathbb{P} \\ +\infty & \text{otherwise,} \end{cases}$$

and $\mathcal{M} = \{\mathbb{P} \in \mathcal{P}_Z : \phi^*_G(\mathbb{P}) = 0\}$.

3.2.1. Dynamic and static short-selling constraints

If there are short-selling constraints on the dynamic as well as static part of the trading strategy, that is, $\Theta = \mathbb{R}^I_0 \cap \mathbb{R}^I_+$, and $h(\theta) = \sum_{i \in I} h_i \theta_i$ for prices $h_i \in \mathbb{R}$, it follows by Proposition 3.2 from (3.1) and condition (i) of Proposition 3.1 that $\phi(X) = \max_{\mathbb{P} \in \mathcal{M}} \mathbb{E}^{\mathbb{P}} X$ for all $X \in U_Z$, where $\mathcal{M}$ is the set of all measures $\mathbb{P} \in \mathcal{P}_Z$ such that

a) $S^1, \dots, S^J$ are supermartingales under $\mathbb{P}$

b) $\mathbb{E}^{\mathbb{P}} H_i \leq h_i$ for all $i \in I$.

3.2.2. Supermartingale transport duality

Suppose now that only the dynamic part of the trading strategy is subject to short-selling constraints and $(H_i)_{i \in I}$ consists of all call options $(S^j_t - K)^+$, $j = 1, \dots, J$, $t = 1, \dots, T$, $K \in \mathbb{R}_+$. If arbitrary quantities of $(S^j_t - K)^+$ can be sold and bought at prices $p^{j,-}_{t,K} \leq p^{j,+}_{t,K}$, respectively, condition (3.1) is satisfied provided that $\lim_{K \to +\infty} p^{j,+}_{t,K} = 0$ for all j and t. So, if in addition, (i) of Proposition 3.1 holds, one obtains from Proposition 3.2 that

$$\phi(X) = \max_{\mathbb{P} \in \mathcal{M}} \mathbb{E}^{\mathbb{P}} X \quad \text{for all } X \in U_Z \tag{3.6}$$

where $\mathcal{M}$ consists of the measures $\mathbb{P} \in \mathcal{P}_Z$ satisfying

a) $S^1, \dots, S^J$ are supermartingales under $\mathbb{P}$

b) $p^{j,-}_{t,K} \leq \mathbb{E}^{\mathbb{P}} (S^j_t - K)^+ = \int_K^{+\infty} \mathbb{P}[S^j_t > x] dx \leq p^{j,+}_{t,K}$ for all $j = 1, \dots, J$, $t = 1, \dots, T$ and $K \in \mathbb{R}_+$.

In the special case $p^{j,-}_{t,K} = p^{j,+}_{t,K} = p^j_{t,K} \in \mathbb{R}_+$, condition b) is satisfied if and only if for all j and t, S^j_t under $\mathbb{P}$ is distributed according to a measure ν^j_t satisfying $\int_K^{+\infty} \nu^j_t(x, +\infty) dx = p^j_{t,K}$ for all $K \in \mathbb{R}_+$. In this case, (3.6) becomes a supermartingale version of Kantorovich's transport duality [34].

4. Superhedging with respect to risk measures

In this section we relax the strict superhedging requirement and consider acceptance sets of the form

$$A = \bigcap_{\mathbb{Q} \in \mathcal{Q}} \{X \in B_Z : \rho_{\mathbb{Q}}(X) \leq 0\} + B^+, \tag{4.1}$$

where $\mathcal{Q} \subseteq \mathcal{P}_Z$ is a non-empty set of probabilistic models, and for every $\mathbb{Q} \in \mathcal{Q}$, $\rho_{\mathbb{Q}} : B_Z \to \mathbb{R}$ is a convex risk measure. More specifically, we concentrate on transformed loss risk measures:

$$\rho_{\mathbb{Q}}(X) = \min_{s \in \mathbb{R}} \left(\mathbb{E}^{\mathbb{Q}} l_{\mathbb{Q}}(s - X) - s \right)$$

for loss functions $l_{\mathbb{Q}} : \mathbb{R} \to \mathbb{R}$. Up to a minus sign they coincide with the optimized certainty equivalents of Ben-Tal and Teboulle [5]. For unbounded random variables, they were studied in Section 5 of [15]. We make the following assumptions:

(l1) every $l_{\mathbb{Q}}$ is increasing[2] and convex with $\lim_{x\to\pm\infty}(l_{\mathbb{Q}}(x)-x)=+\infty$

(l2) $\sup_{\mathbb{Q}}\mathbb{E}^{\mathbb{Q}}l_{\mathbb{Q}}(\varphi(Z))<+\infty$ for an increasing function $\varphi:[1,+\infty)\to\mathbb{R}$ satisfying $\lim_{x\to+\infty}\varphi(x)/x=+\infty$

(l3) $l^*_{\mathbb{Q}}(1)=0$ for all $\mathbb{Q}\in\mathcal{Q}$, where $l^*_{\mathbb{Q}}(y)=\sup_{x\in\mathbb{R}}(xy-l_{\mathbb{Q}}(x))$, $y\in\mathbb{R}$.

Then the following holds:

Lemma 4.1. *For every $\mathbb{Q}\in\mathcal{Q}$, $\rho_{\mathbb{Q}}$ is a real-valued convex risk measure on B_Z with $\rho_{\mathbb{Q}}(0)=0$ and dual representation*

$$\rho_{\mathbb{Q}}(X)=\max_{\mathbb{P}\in\mathcal{P}_Z,\,\mathbb{P}\ll\mathbb{Q}}\left(\mathbb{E}^{\mathbb{P}}[-X]-\mathbb{E}^{\mathbb{Q}}\left[l^*_{\mathbb{Q}}\left(\frac{d\mathbb{P}}{d\mathbb{Q}}\right)\right]\right). \tag{4.2}$$

Moreover, the acceptance set A given in (4.1) *is of the form* (2.3) *for a mapping $\alpha:\mathcal{P}_Z\to\mathbb{R}_+\cup\{+\infty\}$ satisfying* (A1) *and* (A2).

4.1. Robust average value at risk

Now assume that Ω is a closed subset of $\mathbb{R}^{JT}_+$ and consider the filtration $\mathcal{F}_t:=\sigma(S^j_s:j=1,\dots,J,\ s\leq t)$ generated by $S^j_t(\omega)=\omega^j_t$. We also suppose $Z\geq 1+\sum_{j,t}S^j_t$. This ensures that S^j_t belongs to C_Z for all j,t.

For a fixed level $0<\lambda\leq 1$ and $\mathbb{Q}$ from a given non-empty subset $\mathcal{Q}\subseteq\mathcal{P}_Z$, consider the average value at risk

$$\mathrm{AVaR}^{\mathbb{Q}}_{\lambda}(X):=\frac{1}{\lambda}\int_0^{\lambda}\mathrm{VaR}^{\mathbb{Q}}_u(X)du,\quad X\in B_Z.$$

It is well-known (see e.g. [27]) that it can be written as

$$\mathrm{AVaR}^{\mathbb{Q}}_{\lambda}(X)=\min_{s\in\mathbb{R}}\left(\frac{\mathbb{E}^{\mathbb{Q}}(s-X)^+}{\lambda}-s\right),\quad X\in B_Z,$$

and has a dual representation of the form

$$\mathrm{AVaR}^{\mathbb{Q}}_{\lambda}(X)=\max_{\mathbb{P}\ll\mathbb{Q},\,d\mathbb{P}/d\mathbb{Q}\leq 1/\lambda}\mathbb{E}^{\mathbb{P}}[-X],\quad X\in B_Z.$$

The corresponding robust acceptance set is

$$A=\bigcap_{\mathbb{Q}\in\mathcal{Q}}\left\{X\in B_Z:\mathrm{AVaR}^{\mathbb{Q}}_{\lambda}(X)\leq 0\right\}+B^+$$

Let us assume that the assets $S^1,\dots,S^J$ can be traded dynamically with proportional transaction costs given by $\varepsilon_1,\dots,\varepsilon_J\geq 0$, and there exists a family of derivatives with payoffs $(H_i)_{i\in I}\subseteq C_Z$, which can be traded statically with bid and ask prices $h^-_i,h^+_i\in\mathbb{R}$. The resulting set of trading gains G consists of outcomes of the form

$$\sum_{t=1}^{T}\sum_{j=1}^{J}(\vartheta^j_t\Delta S^j_t-\varepsilon_j S^j_{t-1}|\Delta\vartheta^j_t|)+\sum_{i\in I}(\theta_i H_i-\theta^+_i h^+_i+\theta^-_i h^-_i), \tag{4.3}$$

where (ϑ_t) is a J-dimensional $(\mathcal{F}_t)$-predictable strategy and $\theta\in\mathbb{R}^I_0$. Under these conditions, one has the following:

[2] i.e., $l_{\mathbb{Q}}(x)\geq l_{\mathbb{Q}}(y)$ for $x\geq y$

Proposition 4.2. *If $\mathcal{Q}$ is convex, $\sigma(\mathcal{P}_Z, C_Z)$-closed and satisfies*

$$\sup_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}^{\mathbb{Q}}\varphi(Z) < +\infty \text{ for an increasing function } \varphi : [1,+\infty) \to \mathbb{R} \text{ such that } \lim_{x\to+\infty} \frac{\varphi(x)}{x} = +\infty, \tag{4.4}$$

the following are equivalent:

(i) $\mathbb{R}_+ \cap (G - A) = \{0\}$

(ii) *there exists no $X \in G - A$ such that $X(\omega) > 0$ for all $\omega \in \Omega$*

(iii) $\mathcal{M} \neq \emptyset$

(iv) *ϕ is real-valued on B_Z and $\phi(X) = \max_{\mathbb{P}\in\mathcal{M}} \mathbb{E}^{\mathbb{P}} X$ for all $X \in U_Z$,*

where $\mathcal{M}$ is the set of all probability measures $\mathbb{P} \in \mathcal{P}_Z$ satisfying

a) $(1-\varepsilon_j)S^j_{t-1} \leq \mathbb{E}^{\mathbb{P}}[S^j_T \mid \mathcal{F}_{t-1}] \leq (1+\varepsilon_j)S^j_{t-1}$ *for all $j = 1,\dots,J$ and $t = 1,\dots,T$*

b) $h^-_i \leq \mathbb{E}^{\mathbb{P}} H_i \leq h^+_i$ *for all $i \in I$*

c) $d\mathbb{P}/d\mathbb{Q} \leq 1/\lambda$ *for some $\mathbb{Q} \in \mathcal{Q}$.*

Examples 4.3. If $Z = 1 + \sum_{t,j} S^j_t$, the integrability condition (4.4) is satisfied by the following four families of probability measures:
1. All $\mathbb{Q} \in \mathcal{P}_Z$ such that $c^j_t \leq \mathbb{E}^{\mathbb{Q}}(S^j_t)^2 \leq C^j_t$ for given constants $0 \leq c^j_t \leq C^j_t$.
2. All $\mathbb{Q} \in \mathcal{P}_Z$ such that $c^j_t \leq \mathbb{E}^{\mathbb{Q}}[(S^j_t/S^j_{t-1} - 1)^2 \mid \mathcal{F}_{t-1}] \leq C^j_t$ for given constants $0 \leq c^j_t \leq C^j_t$.
3. All $\mathbb{Q} \in \mathcal{P}_Z$ under which $Y^j_t = \log(S^j_t/S^j_{t-1})$, $j = 1,\dots,J$, $t = 1,\dots,T$, forms a Gaussian family with mean vector $(\mathbb{E}^{\mathbb{Q}} Y^j_t)$ in a bounded set $M \subseteq \mathbb{R}^{JT}$ and covariance matrix $\mathrm{Cov}^{\mathbb{Q}}(Y^j_t, Y^k_s)$ in a bounded set $\Sigma \subseteq \mathbb{R}^{JT\times JT}$.
4. The $\sigma(\mathcal{P}_Z, C_Z)$-closed convex hull of any set $\mathcal{Q} \subseteq \mathcal{P}_Z$ satisfying (4.4).

It can easily be checked that the first two families are convex and $\sigma(\mathcal{P}_Z, C_Z)$-closed. But the third one is in general not convex. So to satisfy the assumptions of Proposition 4.2, one has to pass to the $\sigma(\mathcal{P}_Z, C_Z)$-closed convex hull.

4.2. Robust entropic risk measure

As above, suppose that Ω is a closed subset of $\mathbb{R}^{JT}_+$, consider the filtration $(\mathcal{F}_t)$ generated by (S^j_t), $j = 1,\dots,J$, and assume $Z \geq 1 + \sum_{j,t} S^j_t$.

For a fixed risk aversion parameter $\lambda > 0$ and $\mathbb{Q}$ in a given non-empty set $\mathcal{Q} \subseteq \mathcal{P}$, consider the entropic risk measure

$$\mathrm{Ent}^{\mathbb{Q}}_\lambda(X) = \frac{1}{\lambda} \log \mathbb{E}^{\mathbb{Q}} \exp(-\lambda X), \quad X \in B_Z.$$

It admits the alternative representations

$$\mathrm{Ent}^{\mathbb{Q}}_\lambda(X) = \min_{s\in\mathbb{R}} \left(\frac{\exp(\lambda s - 1 - \lambda X)}{\lambda} - s \right) = \max_{\mathbb{P}\ll\mathbb{Q}} \left(\mathbb{E}^{\mathbb{P}}[-X] - \frac{1}{\lambda} \mathbb{E}^{\mathbb{Q}} \left[\frac{d\mathbb{P}}{d\mathbb{Q}} \log \frac{d\mathbb{P}}{d\mathbb{Q}} \right] \right), \quad X \in B_Z;$$

see e.g. [15]. The resulting robust acceptance set is

$$A = \bigcap_{\mathbb{Q} \in \mathcal{Q}} \left\{ X \in B_Z : \mathrm{Ent}_\lambda^{\mathbb{Q}}(X) \le 0 \right\} + B^+.$$

If the set of trading gains G is the same as in (4.3), one obtains the following variant of Proposition 4.2:

Proposition 4.4. *If $\mathcal{Q}$ is convex, $\sigma(\mathcal{P}_Z, C_Z)$-closed and satisfies*

$$\sup_{\mathbb{Q} \in \mathcal{Q}} \mathbb{E}^{\mathbb{Q}} \exp(\varphi(Z)) < +\infty \text{ for an increasing function } \varphi : [1, +\infty) \to \mathbb{R} \text{ with } \lim_{x \to +\infty} \frac{\varphi(x)}{x} = +\infty, \tag{4.5}$$

the following are equivalent:

(i) $\mathbb{R}_+ \cap (G - A) = \{0\}$

(ii) *there exists no $X \in G - A$ such that $X(\omega) > 0$ for all $\omega \in \Omega$*

(iii) *there exists a $\mathbb{P} \in \mathcal{P}_Z$ such that $\mathbb{E}^{\mathbb{P}} X \le 0$ for all $X \in U_Z \cap (G - A)$*

(iv) *ϕ is real-valued on B_Z with $\phi(0) = 0$ and $\phi(X) = \max_{\mathbb{P} \in \mathcal{P}_Z} (\mathbb{E}^{\mathbb{P}} X - \eta(\mathbb{P}))$ for all $X \in U_Z$,*

where $\eta : \mathcal{P}_Z \to \mathbb{R}_+ \cup \{+\infty\}$ is given by

$$\eta(\mathbb{P}) = \begin{cases} \inf_{\mathbb{Q} \in \mathcal{Q},\, \mathbb{P} \ll \mathbb{Q}} \mathbb{E}^{\mathbb{Q}} \left(\frac{d\mathbb{P}}{d\mathbb{Q}} \log \frac{d\mathbb{P}}{d\mathbb{Q}} \right) / \lambda & \text{if } \mathbb{P} \text{ satisfies a)–b) and } \mathbb{P} \ll \mathbb{Q} \text{ for some } \mathbb{Q} \in \mathcal{Q} \\ +\infty & \text{otherwise,} \end{cases}$$

and a)–b) *are the same conditions as in Proposition 4.2.*

Examples 4.5. For $Z = 1 + \sum_{j,t} S_t^j$, the following are convex $\sigma(\mathcal{P}_Z, C_Z)$-closed subsets of $\mathcal{P}_Z$ satisfying (4.5): **1.** All $\mathbb{Q} \in \mathcal{P}_Z$ such that $c_t^j \le \mathbb{E}^{\mathbb{Q}} (S_t^j)^2 \le C_t^j$ and $\mathbb{E}^{\mathbb{Q}} \exp(\varepsilon_t^j (S_T^j)^2) \le D_t^j$ for given constants $0 \le c_t^j \le C_t^j$ and $\varepsilon_t^j, D_t^j > 0$.
2. All $\mathbb{Q} \in \mathcal{P}_Z$ such that $c_t^j \le \mathbb{E}^{\mathbb{Q}}[(S_t^j / S_{t-1}^j - 1)^2 \mid \mathcal{F}_{t-1}] \le C_t^j$ and $\mathbb{E}^{\mathbb{Q}} \exp(\varepsilon_t^j (S_T^j)^2) \le D_t^j$ for given constants $0 \le c_t^j \le C_t^j$ and $\varepsilon_t^j, D_t^j > 0$.
3. The $\sigma(\mathcal{P}_Z, C_Z)$-closed convex hull of any set $\mathcal{Q} \subseteq \mathcal{P}_Z$ satisfying (4.5).

Note that Example 4.3.3 also satisfies condition (4.5). Therefore, its $\sigma(\mathcal{P}_Z, C_Z)$-closed convex hull fulfills the assumptions of Proposition 4.4.

A. Proofs

A.1. Representation of increasing convex functionals

In preparation for the proof of Theorem 2.1 we first derive representation results for general increasing convex functionals on C_Z and U_Z. For a discussion of related representation results based on the Daniell–Stone theorem we refer to [14] and the references therein. As in Section 2, Ω is a non-empty closed subset of $\mathbb{R}^{JT}$ and $Z : \Omega \to [1, +\infty)$ a continuous function such that $\{\omega \in \Omega : Z(\omega) \le z\}$ is bounded for all $z \in \mathbb{R}_+$. If (X_n) is a sequence of functions $X_n : \Omega \to \mathbb{R}$ decreasing pointwise to a function $X:\Omega\to\mathbb{R}$, we write $X_n \downarrow X$. The space C_Z of continuous functions $X:\Omega\to\mathbb{R}$ such that X/Z is bounded is a Stone vector lattice; that is, it is a linear space with the property that for all $X,Y\in C_Z$, the point-wise minima $X\wedge Y$ and $X\wedge 1$ also belong to C_Z. Let ca_Z^+ be the set of all Borel measures μ satisfying $\langle Z,\mu\rangle := \int Z d\mu < +\infty$. We call a functional $\psi: C_Z\to\mathbb{R}$ increasing if $\psi(X)\ge\psi(Y)$ for $X\ge Y$ and define the convex conjugate $\psi^*_{C_Z}: ca_Z^+\to\mathbb{R}\cup\{+\infty\}$ by

$$\psi^*_{C_Z}(\mu) := \sup_{X\in C_Z}\left(\langle X,\mu\rangle - \psi(X)\right).$$

Theorem A.1. *Let $\psi: C_Z\to\mathbb{R}$ be an increasing convex functional with the property that for every $X\in C_Z$ there exists a constant $\varepsilon>0$ such that*

$$\lim_{z\to+\infty}\psi(X+\varepsilon(Z-z)^+) = \psi(X). \tag{A.1}$$

Then

$$\psi(X) = \max_{\mu\in ca_Z^+}\left(\langle X,\mu\rangle - \psi^*_{C_Z}(\mu)\right) \qquad \textit{for all } X\in C_Z. \tag{A.2}$$

Proof. Fix $X\in C_Z$. It is immediate from the definition of $\psi^*_{C_Z}$ that

$$\psi(X)\ge \sup_{\mu\in ca_Z^+}\left(\langle X,\mu\rangle - \psi^*_{C_Z}(\mu)\right). \tag{A.3}$$

On the other hand, it follows from the Hahn–Banach extension theorem that there exists a positive linear functional ζ_X on C_Z such that

$$\zeta_X(Y)\le\psi_X(Y) := \psi(X+Y)-\psi(X) \quad \text{for all } Y\in C_Z.$$

Let (X_n) be a sequence in C_Z satisfying $X_n\downarrow 0$. If we can show that there exists a constant $\eta>0$ such that $\psi_X(\eta X_n)\downarrow 0$, then $\zeta_X(X_n)\downarrow 0$, and we obtain from the Daniell–Stone theorem that ζ_X is of the form $\zeta_X(Y) = \langle Y,\mu_X\rangle$ for a measure $\mu_X\in ca_Z^+$. As a result, one obtains $\psi^*_{C_Z}(\mu_X) = \langle X,\mu_X\rangle - \psi(X)$ and the representation (A.2) follows from (A.3).

Choose $\varepsilon>0$ such that (A.1) holds and $m>0$ so that $X_1\le mZ$. Set $\eta=\varepsilon/(4m)$ and fix $\delta>0$. Let $z\in\mathbb{R}_+$ such that $\psi_X(\varepsilon(Z-z)^+)\le\delta$. By assumption, the set $\Lambda=\{Z\le 2z\}$ is compact. Therefore, one obtains from Dini's lemma that

$$x_n := \max_{\omega\in\Lambda} X_n(\omega)\downarrow 0.$$

Since $x\mapsto\psi_X(x)$ is a convex function from $\mathbb{R}$ to $\mathbb{R}$, it is continuous. In particular, there exists an n_0 such that $\psi_X(2\eta x_n)\le\delta$ for all $n\ge n_0$. Moreover, it follows from

$$X_n\le X_n 1_{\{Z\le 2z\}} + X_1 1_{\{Z>2z\}} \le x_n 1_{\{Z\le 2z\}} + mZ 1_{\{Z>2z\}} \le x_n + 2m(Z-z)^+,$$

that

$$\frac{X_n-x_n}{2m}\le (Z-z)^+,$$

and therefore,

$$\psi_X\left(2\eta(X_n-x_n)\right) = \psi_X\left(\varepsilon\frac{X_n-x_n}{2m}\right)\le\delta \quad \text{for all } n.$$

This gives

$$\psi_X(\eta X_n)\le\frac{\psi_X(2\eta x_n)+\psi_X(2\eta(X_n-x_n))}{2}\le\delta \quad \text{for all } n\ge n_0.$$

Hence, $\psi_X(\eta X_n)\downarrow 0$, and the proof is complete. □

To extend the representation (A.2) beyond C_Z, we need a slightly stronger condition than (A.1).

Lemma A.2. *An increasing convex functional $\psi : C_Z \to \mathbb{R}$ satisfies condition* (A.1) *if*

$$\lim_{z \to +\infty} \psi(n(Z - z)^+) = \psi(0) \quad \textit{for every } n \in \mathbb{N}. \tag{A.4}$$

Proof. If (A.4) holds, one has for any $X \in C_Z$, $\varepsilon \in \mathbb{R}_+$ and $\lambda \in (0, 1)$,

$$\psi(X + \varepsilon(Z - z)^+) \leq \lambda\psi\left(\frac{X}{\lambda}\right) + (1 - \lambda)\psi\left(\varepsilon\frac{(Z - z)^+}{1 - \lambda}\right)$$

as well as

$$\psi\left(\varepsilon\frac{(Z - z)^+}{1 - \lambda}\right) \to \psi(0) \quad \text{for } z \to +\infty.$$

Moreover, since $z \mapsto \psi(zX)$ is a real-valued convex function on $\mathbb{R}$, it is continuous. In particular,

$$\lambda\psi\left(\frac{X}{\lambda}\right) \to \psi(X) \quad \text{and} \quad (1 - \lambda)\psi(0) \to 0 \quad \text{for } \lambda \to 1.$$

This shows that $\psi(X + \varepsilon(Z - z)^+) \to \psi(X)$ for $z \to +\infty$. □

We also need the following

Lemma A.3. *For every $X \in U_Z$ there exists a sequence (X_n) in C_Z such that $X_n \downarrow X$.*

Proof. For $X \in U_Z$, define

$$\tilde{X}_n(\omega) := \sup_{\omega' \in \Omega} \left(\frac{X(\omega')}{Z(\omega')} - n\sum_{j,t} |\omega_t^j - \omega_t'^j|\right).$$

Then $X_n = Z\tilde{X}_n$ is a sequence in C_Z such that $X_n \downarrow X$. □

The next results gives conditions under which a representation of the form (A.2) holds on the set U_Z of all upper semicontinuous functions $X : \Omega \to \mathbb{R}$ such that X/Z is bounded. For $\mu \in ca_Z^+$, we define

$$\psi^*_{U_Z}(\mu) := \sup_{X \in U_Z} (\langle X, \mu\rangle - \phi(X)).$$

Theorem A.4. *Let $\psi : U_Z \to \mathbb{R}$ be an increasing convex functional satisfying condition* (A.4)*. Then the following are equivalent:*

(i) $\psi(X) = \max_{\mu \in ca_Z^+}(\langle X, \mu\rangle - \psi^*_{C_Z}(\mu))$ *for all $X \in U_Z$*

(ii) $\psi(X_n) \downarrow \psi(X)$ *for all $X \in U_Z$ and every sequence (X_n) in C_Z such that $X_n \downarrow X$*

(iii) $\psi(X) = \inf_{Y \in C_Z, Y \geq X} \psi(Y)$ *for all $X \in U_Z$*

(iv) $\psi^*_{C_Z}(\mu) = \psi^*_{U_Z}(\mu)$ *for all $\mu \in ca_Z^+$.*

Proof. Since, by Lemma A.2, (A.4) implies (A.1), we obtain from Theorem A.1 that

$$\psi(X) = \max_{\mu \in ca_Z^+} \big(\langle X, \mu\rangle - \psi^*_{C_Z}(\mu)\big) \quad \text{for all } X \in C_Z.$$

Moreover, by Lemma A.5 below, the sublevel sets $\{\mu \in ca_Z^+ : \psi^*_{C_Z}(\mu) \le a\}$, $a \in \mathbb{R}$, are $\sigma(ca_Z^+, C_Z)$-compact, and there exists a function $\varphi : \mathbb{R}_+ \to \mathbb{R} \cup \{+\infty\}$ such that $\lim_{x \to +\infty} \varphi(x)/x = +\infty$ and $\psi^*_{C_Z}(\mu) \ge \varphi(\langle Z, \mu\rangle)$ for all $\mu \in ca_Z^+$.

(i) $\Rightarrow$ (ii) is now a consequence of part (iii) of Lemma A.6 below, and (ii) $\Rightarrow$ (iii) follows since by Lemma A.3, there exists for every $X \in U_Z$ a sequence (X_n) in C_Z such that $X_n \downarrow X$.

(iii) $\Rightarrow$ (iv): One obviously has $\psi^*_{U_Z} \ge \psi^*_{C_Z}$. On the other hand, if for every $X \in U_Z$, there is a sequence (X_n) in C_Z such that $X_n \ge X$ and $\psi(X_n) \downarrow \psi(X)$, then

$$\sup_n (\langle X_n, \mu\rangle - \psi(X_n)) \ge \langle X, \mu\rangle - \psi(X),$$

from which one obtains $\psi^*_{C_Z} \ge \psi^*_{U_Z}$.

(iv) $\Rightarrow$ (i): For given $X \in U_Z$, one obtains from the definition of $\psi^*_{U_Z}$ that

$$\psi(X) \ge \sup_{\mu \in ca_Z^+} (\langle X, \mu\rangle - \psi^*_{U_Z}(\mu)) = \sup_{\mu \in ca_Z^+} (\langle X, \mu\rangle - \psi^*_{C_Z}(\mu)).$$

Conversely, we know from Lemma A.3 that there exists a sequence (X_n) in C_Z such that $X_n \downarrow X$. So we can conclude by (iii) of Lemma A.6 below that

$$\psi(X) \le \inf_n \psi(X_n) = \max_{\mu \in ca_Z^+} (\langle X, \mu\rangle - \psi^*_{C_Z}(\mu)).$$

□

Lemma A.5. *For every increasing convex functional $\psi : C_Z \to \mathbb{R}$ the following hold:*

(i) *There exists an increasing convex function $\varphi : \mathbb{R}_+ \to \mathbb{R} \cup \{+\infty\}$ satisfying $\lim_{x \to +\infty} \varphi(x)/x = +\infty$ such that $\psi^*_{C_Z}(\mu) \ge \varphi(\langle Z, \mu\rangle)$ for all $\mu \in ca_Z^+$.*

(ii) *If ψ satisfies* (A.4), *the sublevel sets $\{\mu \in ca_Z^+ : \psi^*_{C_Z}(\mu) \le a\}$, $a \in \mathbb{R}$, are $\sigma(ca_Z^+, C_Z)$-compact.*

Proof. For every $\mu \in ca_Z^+$, one has

$$\psi^*_{C_Z}(\mu) \ge \sup_{y \in \mathbb{R}_+} (\langle yZ, \mu\rangle - \psi(yZ)) = \varphi(\langle Z, \mu\rangle)$$

for the increasing convex function $\varphi : \mathbb{R}_+ \to \mathbb{R} \cup \{+\infty\}$ given by

$$\varphi(x) := \sup_{y \in \mathbb{R}_+} (xy - \psi(yZ)).$$

It follows from the fact that ψ is real-valued that $\lim_{x \to +\infty} \varphi(x)/x = +\infty$. This shows (i).

Since $\psi^*_{C_Z}$ is $\sigma(ca_Z^+, C_Z)$-lower semicontinuous, the sets $\Lambda_a := \{\mu \in ca_Z^+ : \psi^*_{C_Z}(\mu) \le a\}$ are $\sigma(ca_Z^+, C_Z)$-closed. If (A.4) holds, every $\mu \in \Lambda_a$ satisfies

$$m \left\langle (Z-z)^+, \mu \right\rangle - \psi(m(Z-z)^+) \le \psi^*_{C_Z}(\mu) \le a \quad \text{for all } m, z \in \mathbb{R}_+.$$

So for given $m \in \mathbb{R}_+$, there exists a $z \in \mathbb{R}_+$ such that

$$\left\langle (Z-z)^+, \mu \right\rangle \leq \frac{a+\psi(0)+1}{m}.$$

In particular,

$$\lim_{z \to +\infty} \sup_{\mu \in \Lambda_a} \left\langle (Z-z)^+, \mu \right\rangle = 0,$$

and, as a result,

$$\lim_{z \to +\infty} \sup_{\mu \in \Lambda_a} \left\langle Z 1_{\{Z>2z\}}, \mu \right\rangle \leq \lim_{z \to +\infty} \sup_{\mu \in \Lambda_a} \left\langle 2(Z-z)^+, \mu \right\rangle = 0. \tag{A.5}$$

From (i) we know that

$$\langle Z, \mu \rangle \leq \varphi^{-1}(a) < +\infty \quad \text{for all } \mu \in \Lambda_a, \tag{A.6}$$

where $\varphi^{-1} : \mathbb{R} \to \mathbb{R}_+$ is the right-continuous inverse of φ given by

$$\varphi^{-1}(y) := \sup \{x \in \mathbb{R}_+ : \varphi(x) \leq y\} \quad \text{with } \sup \emptyset := 0.$$

The mapping $f : X \mapsto X/Z$ identifies C_Z with the space of bounded continuous functions C_b, and $g : \mu \mapsto Z d\mu$ identifies ca^+_Z with the set of all finite Borel measures ca^+. It follows from (A.5) and (A.6) that $g(\Lambda_a)$ is tight. So one obtains from Prokhorov's theorem that $g(\Lambda_a)$ is $\sigma(ca^+, C_b)$-compact, which is equivalent to Λ_a being $\sigma(ca^+_Z, C_Z)$-compact. □

Lemma A.6. *Let $\alpha : ca^+_Z \to \mathbb{R} \cup \{+\infty\}$ be a mapping such that $\inf_{\mu \in ca^+_Z} \alpha(\mu) \in \mathbb{R}$ and $\alpha(\mu) \geq \varphi(\langle Z, \mu \rangle)$ for all $\mu \in ca^+_Z$ and a function $\varphi : \mathbb{R}_+ \to \mathbb{R} \cup \{+\infty\}$ satisfying $\lim_{x \to +\infty} \varphi(x)/x = +\infty$. Then the following hold:*

(i) *$\psi(X) := \sup_{\mu \in ca^+_Z} (\langle X, \mu \rangle - \alpha(\mu))$ defines an increasing convex functional $\psi : B_Z \to \mathbb{R}$.*

(ii) *If all sublevel sets $\left\{\mu \in ca^+_Z : \alpha(\mu) \leq a\right\}$, $a \in \mathbb{R}$, are relatively $\sigma(ca^+_Z, C_Z)$-compact, then $\psi(X_n) \downarrow \psi(X)$ for every sequence (X_n) in C_Z such that $X_n \downarrow X$ for an $X \in C_Z$.*

(iii) *If all sublevel sets $\left\{\mu \in ca^+_Z : \alpha(\mu) \leq a\right\}$, $a \in \mathbb{R}$, are $\sigma(ca^+_Z, C_Z)$-compact, then $\psi(X_n) \downarrow \psi(X)$ for every sequence (X_n) in C_Z such that $X_n \downarrow X$ for an $X \in U_Z$, and*

$$\psi(X) = \max_{\mu \in ca^+_Z} (\langle X, \mu \rangle - \alpha(\mu)) \quad \textit{for all } X \in U_Z.$$

Proof. For every $X \in B_Z$, there exists an $m \in \mathbb{R}$ such that $|X| \leq mZ$. Therefore,

$$\psi(X) \geq \sup_{\mu \in ca^+_Z} (-m \langle Z, \mu \rangle - \alpha(\mu)) > -\infty$$

as well as

$$\psi(X) \leq \sup_{\mu \in ca^+_Z} (m \langle Z, \mu \rangle - \alpha(\mu)) \leq \sup_{\mu \in ca^+_Z} (m \langle Z, \mu \rangle - \varphi(\langle Z, \mu \rangle)) < +\infty.$$

That ψ is increasing and convex is clear. So (i) holds.

Now, let (X_n) be a sequence in C_Z such that $X_n \downarrow X$ for some $X \in U_Z$. By replacing φ with

$$\tilde{\varphi}(x) = \inf_{y \geq x} \varphi(y) \vee \inf_{\mu \in ca_Z^+} \alpha(\mu),$$

one can assume that φ is increasing. Then the right-continuous inverse $\varphi^{-1} : \mathbb{R} \to \mathbb{R}_+$ given by

$$\varphi^{-1}(y) := \sup \{x \in \mathbb{R}_+ : \varphi(x) \leq y\} \quad \text{with } \sup \emptyset := 0,$$

satisfies $\lim_{y \to +\infty} \varphi^{-1}(y)/y = 0$ and $\langle Z, \mu \rangle \leq \varphi^{-1}(\alpha(\mu))$ for all

$$\mu \in \operatorname{dom} \alpha := \left\{ \nu \in ca_Z^+ : \alpha(\nu) < +\infty \right\}.$$

Choose $m \in \mathbb{R}_+$ such that $X_1 \leq mZ$. Then

$$\langle X_n, \mu \rangle - \alpha(\mu) \leq m \langle Z, \mu \rangle - \alpha(\mu) \leq m\varphi^{-1}(\alpha(\mu)) - \alpha(\mu) \quad \text{for all } n \text{ and } \mu \in \operatorname{dom} \alpha.$$

If the sets $\left\{ \mu \in ca_Z^+ : \alpha(\mu) \leq a \right\}$, $a \in \mathbb{R}$, are $\sigma(\mathcal{P}_Z, C_Z)$-compact, α is $\sigma(\mathcal{P}_Z, C_Z)$-lower secicontinuous, and it follows that for $a \in \mathbb{R}$ large enough, there exists a sequence (μ_n) in $\left\{ \mu \in ca_Z^+ : \alpha(\mu) \leq a \right\}$ such that

$$\psi(X_n) = \langle X_n, \mu_n \rangle - \alpha(\mu_n) \quad \text{for all } n.$$

Since $\sigma(ca^+, C_b)$ is metrizable, the same is true for $\sigma(ca_Z^+, C_Z)$. Therefore, after possibly passing to a subsequence, one can assume that (μ_n) converges to a measure $\mu \in \left\{ \mu \in ca_Z^+ : \alpha(\mu) \leq a \right\}$ in $\sigma(ca_Z^+, C_Z)$. Then

$$\alpha(\mu) \leq \liminf_n \alpha(\mu_n).$$

Moreover, for every $\varepsilon > 0$, there is an n' such that $\langle X_{n'}, \mu \rangle \leq \langle X, \mu \rangle + \varepsilon$. Now choose $n \geq n'$ such that $\langle X_{n'}, \mu_n \rangle \leq \langle X_{n'}, \mu \rangle + \varepsilon$. Then

$$\langle X_n, \mu_n \rangle \leq \langle X_{n'}, \mu_n \rangle \leq \langle X_{n'}, \mu \rangle + \varepsilon \leq \langle X, \mu \rangle + 2\varepsilon$$

showing that, $\limsup_n \langle X_n, \mu_n \rangle \leq \langle X, \mu \rangle$, and therefore,

$$\inf_n \psi(X_n) \leq \limsup_n \left(\langle X_n, \mu_n \rangle - \alpha(\mu_n) \right) \leq \langle X, \mu \rangle - \alpha(\mu) \leq \psi(X).$$

In particular,

$$\psi(X_n) \downarrow \psi(X) = \max_{\mu \in ca_Z^+} \left(\langle X, \mu \rangle - \alpha(\mu) \right).$$

Since by Lemma A.3, every function $X \in U_Z$ can be appromixated from above by a decreasing sequence (X_n) in C_Z, this shows (iii).

It remains to prove (ii). To do that we note that if α satisfies the assumption of (ii), the $\sigma(ca_Z^+, C_Z)$-lower semicontinuous hull α_* has $\sigma(ca_Z^+, C_Z)$-compact sublevel sets and

$$\alpha_*(\mu) \geq \varphi_*(\langle Z, \mu \rangle) \vee \inf_{\mu \in ca_Z^+} \alpha(\mu) \quad \text{for all } \mu \in ca_Z^+,$$

where φ_* is the lower semicontinuous hull of φ. Since $\lim_{x \to +\infty} \varphi_*(x)/x = +\infty$ and

$$\psi(X) = \sup_{\mu \in ca_Z^+} \left(\langle X, \mu \rangle - \alpha_*(\mu) \right), \quad \text{for all } X \in C_Z,$$

it follows from (iii) that $\psi(X_n) \downarrow \psi(X)$ for every sequence (X_n) in C_Z such that $X_n \downarrow X$ for an $X \in C_Z$. This shows (ii). □

A.2. Proof of Theorem 2.1

Having Theorems A.1 and A.4 in hand, we now are ready to prove Theorem 2.1. That (iv) implies (i) is clear, and the implications (iii) ⇒ (ii) ⇒ (i) as well as (vi) ⇒ (v) ⇒ (iv) ⇒ (i) can be shown without assumptions:

(iii) ⇒ (ii): It follows from (iii) that

$$\min_{\mathbb{P}\in\mathcal{P}_Z}\phi^*(\mathbb{P}) = -\max_{\mathbb{P}\in\mathcal{P}_Z} -\phi^*(\mathbb{P}) = -\phi(0) = 0.$$

In particular, since $\phi(X)\leq 0$ for all $X\in G-A$, there exists a $\mathbb{P}\in\mathcal{P}_Z$ such that

$$0=\phi^*(\mathbb{P}) = \sup_{X\in C_Z}\left(\mathbb{E}^{\mathbb{P}}X-\phi(X)\right)\geq \sup_{X\in C_Z\cap(G-A)}\left(\mathbb{E}^{\mathbb{P}}X-\phi(X)\right)\geq \sup_{X\in C_Z\cap(G-A)}\mathbb{E}^{\mathbb{P}}X,$$

and therefore, $\mathbb{E}^{\mathbb{P}}X\leq 0$ for all $X\in C_Z\cap(G-A)$.

(ii) ⇒ (i): By our assumptions on A and G, $G-A$ contains 0. On the other hand, if (ii) holds, there exists a $\mathbb{P}\in\mathcal{P}_Z$ such that $\mathbb{E}^{\mathbb{P}}m\leq 0$ for all $m\in\mathbb{R}\cap(G-A)$, implying that $\mathbb{R}_+\cap(G-A)=\{0\}$.

(vi) ⇒ (v): If (vi) holds, then $\phi^*(\mathbb{P})\geq \sup_{X\in U_Z}(\mathbb{E}^{\mathbb{P}}X-\phi(X))$ for all $\mathbb{P}\in\mathcal{P}_Z$. So (v) follows from (vi) like (ii) from (iii).

(v) ⇒ (iv): Assume there exists an $X\in G-A$ such that $X(\omega)>0$ for all $\omega\in\Omega$ and fix a $\mathbb{P}\in\mathcal{P}_Z$. Since $\mathbb{P}$ is regular, there exist an $\varepsilon>0$ and a closed set $E\subseteq\{X\geq\varepsilon\}$ such that $\mathbb{P}[E]>0$, or equivalently, $\mathbb{E}^{\mathbb{P}}1_E>0$. But since $\varepsilon 1_E$ belongs to $U_Z\cap(G-A)$, this contradicts (v).

Now we assume (2.1) and show (i) ⇒ (iii). Since $A-G$ is convex and contains B^+, the mapping $\phi: B_Z\to\mathbb{R}\cup\{\pm\infty\}$ is increasing and convex. Moreover, it follows from (i) that $\phi(m)=m$ for all $m\in\mathbb{R}$, from which one obtains $\phi(X)\in\mathbb{R}$ for every bounded $X\in B$. By condition (2.1), there exists for every $n\in\mathbb{N}$ a $z\in\mathbb{R}_+$ such that $\phi(n(Z-z)^+)\leq 1/n$, and therefore,

$$\phi\left(\frac{n}{2}Z\right)\leq\frac{\phi(nz)+\phi(n(Z-z)^+)}{2}\leq\frac{nz}{2}+\frac{1}{2n}.$$

So one obtains from monotonicity and convexity that ϕ is real-valued on B_Z. In addition, it follows from Lemma A.2 that ϕ satisfies condition (A.1). Therefore, Theorem A.1 yields $\phi(X)=\max_{\mu\in ca_Z^+}(\langle X,\mu\rangle-\phi^*(\mu))$ for all $X\in C_Z$. Since $\phi(m)=m$ for all $m\in\mathbb{R}$, $\phi^*(\mu)$ is $+\infty$ for all $\mu\in ca_Z^+\setminus\mathcal{P}_Z$, and one obtains $\phi(X)=\max_{\mathbb{P}\in\mathcal{P}_Z}(\mathbb{E}^{\mathbb{P}}X-\phi^*(\mathbb{P}))$ for all $X\in C_Z$.

Finally, if both conditions, (2.1) and (2.2), hold, one obtains from Theorem A.4 that (iii) implies (vi), and the proof of Theorem 2.1 is complete. □

A.3. Proofs of Propositions 2.2 and 2.3

Proof of Proposition 2.2

If the acceptance set A is of the form (2.3) for a mapping $\alpha:\mathcal{P}_Z\to\mathbb{R}_+\cup\{+\infty\}$ satisfying (A1)–(A2), it can be written as

$$A=\{X\in B_Z:\rho(X)\leq 0\}+B^+\quad\text{for}\quad \rho(X):=\sup_{\mathbb{P}\in\mathcal{P}_Z}\left(\mathbb{E}^{\mathbb{P}}(-X)-\alpha(\mathbb{P})\right).$$

By passing to the lower convex hull, one can assume that β is convex. Then, Jensen's inequality yields

$$\alpha(\mathbb{P})\geq\mathbb{E}^{\mathbb{P}}\beta(Z)\geq\beta(\mathbb{E}^{\mathbb{P}}Z)\quad\text{for all }\mathbb{P}\in\mathcal{P}_Z.$$

It follows from Prokhorov's theorem that the sets $\{\mathbb{P} \in \mathcal{P}_Z : \mathbb{E}^{\mathbb{P}}\beta(Z) \le a\}$, $a \in \mathbb{R}$, are $\sigma(\mathcal{P}_Z, C_Z)$-compact. As a consequence, the sets $\{\mathbb{P} \in \mathcal{P}_Z : \alpha(\mathbb{P}) \le a\}$, $a \in \mathbb{R}$, are relatively $\sigma(\mathcal{P}_Z, C_Z)$-compact, and one obtains from part (ii) of Lemma A.6 that for every $n \in \mathbb{N}$,

$$\rho\left(\frac{1}{n} - n(Z-z)^+\right) = -\frac{1}{n} + \rho\left(-n(Z-z)^+\right) \downarrow -\frac{1}{n} \quad \text{as } z \to +\infty.$$

In particular, $1/n - n(Z-z)^+ \in A \subseteq A - G$ for z large enough, showing that condition (2.1) holds. □

Proof of Proposition 2.3

By our assumptions on G and A, one has $\phi(0) \le 0$. If $\phi(0) = -\infty$, $G - A$ contains $\mathbb{R}$, which by (2.1), implies $G - A = B_Z$. Then $\phi \equiv -\infty$, $\phi^* \equiv +\infty$ and all the statements of Proposition 2.3 become obvious. On the other hand, if $\phi(0) > -\infty$, it follows from (2.1) like in the proof of Theorem 2.1, that ϕ is real-valued on B_Z. Then

$$\phi^*(\mathbb{P}) \le \sup_{X \in U_Z} (\mathbb{E}^{\mathbb{P}} X - \phi(X)) \quad \text{for all } \mathbb{P} \in \mathcal{P}_Z,$$

and by Theorem A.4, the inequality is an equality if and only if ϕ satisfies condition (2.2). Next, note that since

$$X - \phi(X) - \varepsilon \in C_Z \cap (G - A) \quad \text{for all } X \in C_Z \text{ and } \varepsilon > 0,$$

one has

$$\phi^*(\mathbb{P}) = \sup_{X \in C_Z} \mathbb{E}^{\mathbb{P}}(X - \phi(X)) = \sup_{X \in C_Z \cap (G-A)} \mathbb{E}^{\mathbb{P}} X,$$

and analogously,

$$\sup_{X \in U_Z} (\mathbb{E}^{\mathbb{P}} X - \phi(X)) = \sup_{X \in U_Z \cap (G-A)} \mathbb{E}^{\mathbb{P}} X.$$

This completes the proof of Proposition 2.3. □

A.4. Proofs of Propositions 3.1 and 3.2

Proof of Proposition 3.1

Let us first show (iv) ⇒ (iii) ⇒ (ii) ⇒ (i). If (iv) holds, one has $0 = \phi(0) = \max_{\mathbb{P} \in \mathcal{P}_Z} -\phi^*_G(\mathbb{P})$, yielding (iii). Moreover, since $\mathbb{E}^{\mathbb{P}} X \le 0$ for every $X \in G$ and $\mathbb{P} \in \mathcal{M}$, (iii) implies (ii). That (ii) implies (i) is clear.

To prove (i) ⇒ (iv), we first note that (3.1) implies (2.1) and (i) is a reformulation of condition (i) of Theorem 2.1 in the case $A = B^+$. So, by Theorem 2.1, it follows from (i) that ϕ is real-valued on B_Z with $\phi(0) = 0$. We know from Proposition 2.3 that

$$\phi^*(\mathbb{P}) \le \sup_{X \in U_Z \cap (G - B^+)} \mathbb{E}^{\mathbb{P}} \le \sup_{X \in G} \mathbb{E}^{\mathbb{P}} X = \phi^*_G(\mathbb{P}), \quad \mathbb{P} \in \mathcal{P}_Z.$$

So if we can show

$$\phi^*(\mathbb{P}) \ge \phi^*_G(\mathbb{P}), \quad \mathbb{P} \in \mathcal{P}_Z, \tag{A.7}$$

we obtain $\phi^* = \phi^*_G$, and by Proposition 2.3, condition (2.2) holds. Then it follows from Theorem 2.1 that (i) implies (iv). To prove (A.7), we observe that

$$\sup_{X \in G} \mathbb{E}^{\mathbb{P}} X = \sup_{\vartheta} \mathbb{E}^{\mathbb{P}}\Big[\sum_{t,j} \vartheta^j_t \Delta S^j_t - g^j_{t-1}(\Delta \vartheta^j_t S^j_{t-1})\Big] + \sup_{\theta}\Big(\sum_i \theta_i \mathbb{E}^{\mathbb{P}} H_i - h(\theta)\Big),$$

and since $g^j_{t-1}(0) = 0$, the first supremum can be taken over strategies ϑ such that

$$\mathbb{E}^{\mathbb{P}}\Big[\sum_{t,j} \vartheta^j_t \Delta S^j_t - g^j_{t-1}(\Delta\vartheta^j_t S^j_{t-1})\Big] \geq 0.$$

Then $\mathbb{E}^{\mathbb{P}}[\sum_{t,j} \vartheta^j_t \Delta S^j_t - g^j_{t-1}(\Delta\vartheta^j_t S^j_{t-1})]$ can be approximated by $\mathbb{E}^{\mathbb{P}}[\sum_{t,j} \tilde{\vartheta}^j_t \Delta S^j_t - g^j_{t-1}(\Delta\tilde{\vartheta}^j_t S^j_{t-1})]$ for continuous $\mathcal{F}_{t-1}$-measurable functions $\tilde{\vartheta}^j_t : \Omega \to \mathbb{R}$ with compact support. Since

$$\sum_{t,j} (\tilde{\vartheta}^j_t \Delta S^j_t) - g^j_{t-1}(\Delta\tilde{\vartheta}^j_t S^j_{t-1})) + \sum_i \theta_i H_i - h(\theta) \in C_Z \cap G,$$

it follows that $\phi^*(\mathbb{P}) \geq \sup_{X \in C_Z \cap G} \mathbb{E}^{\mathbb{P}} X \geq \phi^*_G(\mathbb{P})$ for all $\mathbb{P} \in \mathcal{P}_Z$.

It remains to show that ϕ^*_G is of the form (3.2). To do this we note that

$$\sum_{t=1}^T \vartheta^j_t \Delta S^j_t = \sum_{t=1}^T \sum_{s=1}^t \Delta\vartheta^j_s \Delta S^j_t = \sum_{s=1}^T \sum_{t=s}^T \Delta\vartheta^j_s \Delta S^j_t = \sum_{t=1}^T \Delta\vartheta^j_t (S^j_T - S^j_{t-1}).$$

Hence,

$$\sup_{X \in G} \mathbb{E}^{\mathbb{P}} X = \sup_{\vartheta} \mathbb{E}^{\mathbb{P}}\Big[\sum_{t,j} \Delta\vartheta^j_t (S^j_T - S^j_{t-1}) - g^j_{t-1}(\Delta\vartheta^j_t S^j_{t-1})\Big] + \sup_{\theta} \Big(\sum_i \theta_i \mathbb{E}^{\mathbb{P}} H_i - h(\theta)\Big),$$

where the first supremum can be taken over strategies ϑ such that

$$\mathbb{E}^{\mathbb{P}}\Big[\sum_{t,j} \Delta\vartheta^j_t (S^j_T - S^j_{t-1}) - g^j_{t-1}(\Delta\vartheta^j_t S^j_{t-1})\Big] \geq 0.$$

Now $\mathbb{E}^{\mathbb{P}}\Big[\sum_{t,j} \Delta\vartheta^j_t (S^j_T - S^j_{t-1}) - g^j_{t-1}(\Delta\vartheta^j_t S^j_{t-1})\Big]$ can be approximated by

$$\mathbb{E}^{\mathbb{P}}\Big[\sum_{t,j} \Delta\tilde{\vartheta}^j_t (S^j_T - S^j_{t-1}) - g^j_{t-1}(\Delta\tilde{\vartheta}^j_t S^j_{t-1})\Big] = \sum_{t,j} \mathbb{E}^{\mathbb{P}}\Big[\Delta\tilde{\vartheta}^j_t (\mathbb{E}^{\mathbb{P}}[S^j_T \mid \mathcal{F}_{t-1}] - S^j_{t-1}) - g^j_{t-1}(\Delta\tilde{\vartheta}^j_t S^j_{t-1})\Big]$$

for bounded $\mathcal{F}_{t-1}$-measurable mappings $\Delta\tilde{\vartheta}^j_t$ with compact support. On $\{S^j_{t-1} > 0\}$, one has

$$\begin{aligned}
&\sup_{\Delta\tilde{\vartheta}^j_t} \Big(\Delta\tilde{\vartheta}^j_t (\mathbb{E}^{\mathbb{P}}[S^j_T \mid \mathcal{F}_{t-1}] - S^j_{t-1}) - g^j_{t-1}(\Delta\tilde{\vartheta}^j_t S^j_{t-1})\Big) \\
= &\sup_{\Delta\tilde{\vartheta}^j_t} \left(\Delta\tilde{\vartheta}^j_t S^j_{t-1} \frac{\mathbb{E}^{\mathbb{P}}[S^j_T \mid \mathcal{F}_{t-1}] - S^j_{t-1}}{S^j_{t-1}} - g^j_{t-1}(\Delta\tilde{\vartheta}^j_t S^j_{t-1})\right) \\
= &\; g^{j*}_{t-1}\left(\frac{\mathbb{E}^{\mathbb{P}}[S^j_T \mid \mathcal{F}_{t-1}] - S^j_{t-1}}{S^j_{t-1}}\right),
\end{aligned}$$

and on $\{S^j_{t-1} = 0\}$,

$$\sup_{\Delta\tilde{\vartheta}^j_t} \Big(\Delta\tilde{\vartheta}^j_t (\mathbb{E}^{\mathbb{P}}[S^j_T \mid \mathcal{F}_{t-1}] - S^j_{t-1}) - g^j_{t-1}(\Delta\tilde{\vartheta}^j_t S^j_{t-1})\Big) = \sup_{\Delta\tilde{\vartheta}^j_t} \Delta\tilde{\vartheta}^j_t \mathbb{E}^{\mathbb{P}}[S^j_T \mid \mathcal{F}_{t-1}] = +\infty 1_{\{\mathbb{E}^{\mathbb{P}}[S^j_T \mid \mathcal{F}_{t-1}] > 0\}}.$$

Since $\mathbb{P}[S^j_{t-1} = 0$ and $\mathbb{E}^{\mathbb{P}}[S^j_T > 0 \mid \mathcal{F}_{t-1}] > 0] > 0$ if and only if $\mathbb{P}[S^j_{t-1} = 0$ and $S^j_T > 0] > 0$, this proves (3.2). □

Proof of Proposition 3.2
It follows as in the proof of Proposition 3.1 that

$$\phi^*(\mathbb{P}) = \sup_{X \in U_Z} \left(\mathbb{E}^{\mathbb{P}} X - \phi(X) \right) = \phi^*_G(\mathbb{P}) \quad \text{for all } \mathbb{P} \in \mathcal{P}_Z,$$

and (i)–(iv) are equivalent. Moreover,

$$\phi^*_G(\mathbb{P}) = \sup_{\vartheta \geq 0} \mathbb{E}^{\mathbb{P}} \Big[\sum_{t,j} \vartheta^j_t \Delta S^j_t \Big] + \sup_{\theta \in \Theta} \mathbb{E}^{\mathbb{P}} \left(\sum_{i \in I} \theta_i H_i - h_i \right),$$

and since the first supremum can be taken over non-negative predictable strategies (ϑ_t) such that

$$\mathbb{E}^{\mathbb{P}} \Big[\sum_{t,j} \vartheta^j_t \Delta S^j_t \Big] \geq 0,$$

it can equivalently be taken over bounded non-negative predictable strategies. Now, it is easy to see that

$$\phi^*_G(\mathbb{P}) = \begin{cases} \sup_\theta \mathbb{E}^{\mathbb{P}} (\sum_{i \in I} \theta_i H_i - h(\theta)) & \text{if } S^1, \dots, S^J \text{ are supermartingales under } \mathbb{P} \\ +\infty & \text{otherwise.} \end{cases}$$

□

A.5. Proofs of Lemma 4.1 and Propositions 4.2 and 4.4

Proof of Lemma 4.1
It follows from (l1)–(l2) that for all $X \in B_Z$, $s \mapsto \mathbb{E}^{\mathbb{Q}} l_{\mathbb{Q}}(s - X)$ is a real-valued increasing convex function on $\mathbb{R}$ such that

$$\lim_{s \to \pm\infty} \mathbb{E}^{\mathbb{Q}} l_{\mathbb{Q}}(s - X) - s = +\infty.$$

In particular, there exists a minimizing s, and it is easy to see that $\rho_{\mathbb{Q}}$ is a real-valued decreasing convex functional on B_Z with the translation property $\rho_{\mathbb{Q}}(X + m) = \rho_{\mathbb{Q}}(X) - m$, $m \in \mathbb{R}$. Moreover, $\mathbb{E}^{\mathbb{Q}} l_{\mathbb{Q}}(cZ) < +\infty$ for all $c \in \mathbb{R}_+$. So B_Z is contained in the Orlicz heart $M^{l_{\mathbb{Q}}}$ corresponding to $\mathbb{Q}$ and the Young function $l_{\mathbb{Q}}(.) - l_{\mathbb{Q}}(0)$. By Theorem 4.6 and the computation in Section 5.4 of [15],

$$\rho_{\mathbb{Q}}(X) = \max_{\mathbb{P}} \left(\mathbb{E}^{\mathbb{P}}[-X] - \mathbb{E}^{\mathbb{Q}} \left[l^*_{\mathbb{Q}} \left(\frac{d\mathbb{P}}{d\mathbb{Q}} \right) \right] \right), \quad X \in B_Z,$$

where the maximum is over all $\mathbb{P} \ll \mathbb{Q}$ such that $d\mathbb{P}/d\mathbb{Q}$ is in the norm-dual of $M^{l_{\mathbb{Q}}}$. For all these $\mathbb{P}$, one has $\mathbb{E}^{\mathbb{P}} Z = \mathbb{E}^{\mathbb{Q}}[Z d\mathbb{P}/d\mathbb{Q}] < +\infty$, yielding that $\mathbb{P} \in \mathcal{P}_Z$. On the other hand, since $l_{\mathbb{Q}}(x) \geq xy - l^*_{\mathbb{Q}}(y)$ for all $x, y \in \mathbb{R}$, one has for every $X \in B_Z$, $s \in \mathbb{R}$ and $\mathbb{P} \ll \mathbb{Q}$,

$$\mathbb{E}^{\mathbb{Q}} l_{\mathbb{Q}}(s - X) - s \geq \mathbb{E}^{\mathbb{Q}} \left[(s - X) \frac{d\mathbb{P}}{d\mathbb{Q}} \right] - \mathbb{E}^{\mathbb{Q}} \left[l^*_{\mathbb{Q}} \left(\frac{d\mathbb{P}}{d\mathbb{Q}} \right) \right] - s = \mathbb{E}^{\mathbb{P}}[-X] - \mathbb{E}^{\mathbb{Q}} \left[l^*_{\mathbb{Q}} \left(\frac{d\mathbb{P}}{d\mathbb{Q}} \right) \right].$$

This proves the duality (4.2). By Jensen's inequality and (l3),

$$\mathbb{E}^{\mathbb{Q}} l^*_{\mathbb{Q}}(d\mathbb{P}/d\mathbb{Q}) \geq l^*_{\mathbb{Q}} \mathbb{E}^{\mathbb{Q}}[d\mathbb{P}/d\mathbb{Q}] = 0,$$

and therefore, $\min_{\mathbb{P} \in \mathcal{P}_Z} \mathbb{E}^{\mathbb{Q}} l^*_{\mathbb{Q}}(d\mathbb{P}/d\mathbb{Q}) = \mathbb{E}^{\mathbb{Q}} l^*_{\mathbb{Q}}(d\mathbb{Q}/d\mathbb{Q}) = 0$, showing that $\rho_{\mathbb{Q}}(0) = 0$. Moreover, since

$$\bigcap_{\mathbb{Q} \in \mathcal{Q}} \{X \in B_Z : \rho_{\mathbb{Q}}(X) \leq 0\} = \{X \in B_Z : \sup_{\mathbb{Q} \in \mathcal{Q}} \rho_{\mathbb{Q}}(X) \leq 0\},$$

the acceptance set A can be written as

$$A = \left\{ X \in B_Z : \mathbb{E}^{\mathbb{P}} X + \alpha(\mathbb{P}) \geq 0 \text{ for all } \mathbb{P} \in \mathcal{P}_Z \right\} + B^+,$$

where $\alpha(\mathbb{P}) := \inf_{\mathbb{Q} \in \mathcal{Q}} \alpha_{\mathbb{Q}}(\mathbb{P})$ with

$$\alpha_{\mathbb{Q}}(\mathbb{P}) := \begin{cases} \mathbb{E}^{\mathbb{Q}} l^*_{\mathbb{Q}}(d\mathbb{P}/d\mathbb{Q}) & \text{if } \mathbb{P} \ll \mathbb{Q} \\ +\infty & \text{otherwise.} \end{cases}$$

It follows from $\min_{\mathbb{P} \in \mathcal{P}_Z} \alpha_{\mathbb{Q}}(\mathbb{P}) = 0$ for all $\mathbb{Q} \in \mathcal{Q}$ that $\min_{\mathbb{P} \in \mathcal{P}_Z} \alpha(\mathbb{P}) = 0$. Hence (A1) holds. Moreover, since $l^*_{\mathbb{Q}}(y) \geq xy - l_{\mathbb{Q}}(x)$ for all $x, y \in \mathbb{R}$, one has for every $\mathbb{P} \in \mathcal{P}_Z$ for which there exists a $\mathbb{Q} \in \mathcal{Q}$ such that $\mathbb{P} \ll \mathbb{Q}$,

$$\alpha(\mathbb{P}) = \inf_{\mathbb{Q} \in \mathcal{Q},\, \mathbb{P} \ll \mathbb{Q}} \mathbb{E}^{\mathbb{Q}} l^*_{\mathbb{Q}} \left(\frac{d\mathbb{P}}{d\mathbb{Q}} \right) \geq \inf_{\mathbb{Q} \in \mathcal{Q},\, \mathbb{P} \ll \mathbb{Q}} \mathbb{E}^{\mathbb{Q}} \left[\varphi(Z) \frac{d\mathbb{P}}{d\mathbb{Q}} - l_{\mathbb{Q}}(\varphi(Z)) \right] \geq \mathbb{E}^{\mathbb{P}} \varphi(Z) - \sup_{\mathbb{Q} \in \mathcal{Q}} \mathbb{E}^{\mathbb{Q}} l_{\mathbb{Q}}(\varphi(Z)),$$

which implies (A2) for $\beta = \varphi - \sup_{\mathbb{Q}} \mathbb{E}^{\mathbb{Q}} l_{\mathbb{Q}}(\varphi(Z))$. □

Proof of Proposition 4.2
Since $\mathrm{AVaR}^{\mathbb{Q}}_{\lambda}$ is a transformed loss risk measure with loss function $l_{\mathbb{Q}}(x) = x^+/\lambda$, it follows from the integrability condition (4.4) that the assumptions of Lemma 4.1 are satisfied. So one obtains from Proposition 2.2 that condition (2.1) holds. Moreover, $G - A$ is a convex cone. Therefore, by Proposition 2.3,

$$\phi^*(\mathbb{P}) = \sup_{X \in C_Z \cap (G-A)} \mathbb{E}^{\mathbb{P}} X = \begin{cases} 0 & \text{if } \mathbb{E}^{\mathbb{P}} X \leq \text{ for all } X \in C_Z \cap (G-A) \\ +\infty & \text{otherwise.} \end{cases} \tag{A.8}$$

Let us denote $\hat{\mathcal{M}} := \{\mathbb{P} \in \mathcal{P}_Z : \phi^*(\mathbb{P}) = 0\}$ and write $\mathcal{M} = \mathcal{M}_G \cap \mathcal{M}_A$, where $\mathcal{M}_G$ is the set of all $\mathbb{P} \in \mathcal{P}_Z$ satisfying a)–b) and $\mathcal{M}_A$ the set of all $\mathbb{P} \in \mathcal{P}_Z$ satisfying c). Since for $X \in A$, the negative part X^- belongs to B_Z, one obtains from (A.8),

$$\phi^*(\mathbb{P}) \leq \sup_{X \in U_Z \cap (G-A)} \mathbb{E}^{\mathbb{P}} X \leq \sup_{X \in G,\, \mathbb{E}^{\mathbb{P}} X > -\infty} \mathbb{E}^{\mathbb{P}} X - \inf_{X \in A} \mathbb{E}^{\mathbb{P}} X \quad \text{for all } \mathbb{P} \in \mathcal{P}_Z. \tag{A.9}$$

It follows as in the proof of Proposition 3.1 that the second supremum in (A.9) is zero for all $\mathbb{P} \in \mathcal{M}_G$, while it can be seen from the dual representation

$$\mathrm{AVaR}^{\mathbb{Q}}_{\lambda}(X) = \sup_{\mathbb{P} \in \mathcal{P}_Z,\, d\mathbb{P}/d\mathbb{Q} \leq 1/\lambda} \mathbb{E}^{\mathbb{P}}[-X]$$

that the infimum is zero for all $\mathbb{P} \in \mathcal{M}_A$. In particular, $\hat{\mathcal{M}} \supseteq \mathcal{M} = \mathcal{M}_G \cap \mathcal{M}_A$.

On the other hand, it can be shown as in the proof of Proposition 3.1 that $\hat{\mathcal{M}} \subseteq \mathcal{M}_G$. Moreover, it follows from our assumptions on $\mathcal{Q}$ that $\mathcal{M}_A$ is convex and $\sigma(\mathcal{P}_Z, C_Z)$-closed. Indeed, for $\mathbb{P}_1, \mathbb{P}_2 \in \mathcal{M}_A$ and $0 \le \mu \le 1$, there exist $Y_1, Y_2 \in B^+$ bounded by $1/\lambda$ such that $\mathbb{P}_1 = Y_1 \cdot \mathbb{Q}_1$ and $\mathbb{P}_2 = Y_2 \cdot \mathbb{Q}_2$. Therefore,

$$(\mu \mathbb{P}_1 + (1-\mu)\mathbb{P}_2)[E] \le \frac{1}{\lambda}(\mu \mathbb{Q}_1 + (1-\mu)\mathbb{Q}_2)[E]$$

for all measurable sets E, and it follows that

$$\frac{d(\mu \mathbb{P}_1 + (1-\mu)\mathbb{P}_2)}{d(\mu \mathbb{Q}_1 + (1-\mu)\mathbb{Q}_2)} \le \frac{1}{\lambda},$$

showing that $\mathcal{M}_A$ is convex. Furthermore, if $(\mathbb{P}_n)$ is a sequence in $\mathcal{M}_A$ converging to a $\mathbb{P} \in \mathcal{P}_Z$ in $\sigma(\mathcal{P}_Z, C_Z)$, there exist $\mathbb{Q}_n \in \mathcal{Q}$ and $Y_n \in B^+$ bounded by $1/\lambda$ such that $\mathbb{P}_n = Y_n \cdot \mathbb{Q}_n$. It follows from condition (4.4) that $\mathcal{Q}$ is $\sigma(\mathcal{P}_Z, C_Z)$-compact. So by passing to a subsequence, one can assume that $\mathbb{Q}_n$ converges to a $\mathbb{Q}$ in $\mathcal{Q}$ with respect to $\sigma(\mathcal{P}_Z, C_Z)$. Then

$$\mathbb{E}^{\mathbb{P}} X = \lim_n \mathbb{E}^{\mathbb{P}_n} X \le \frac{1}{\lambda} \lim_n \mathbb{E}^{\mathbb{Q}_n} X = \frac{1}{\lambda} \mathbb{E}^{\mathbb{Q}} X \quad \text{for all } X \in C_Z^+.$$

Since $\mathbb{P}$ and $\mathbb{Q}$ are regular, this implies $d\mathbb{P}/d\mathbb{Q} \le 1/\lambda$ and hence, shows that $\mathcal{M}_A$ is $\sigma(\mathcal{P}_Z, C_Z)$-closed. By a separating hyperplane argument, one obtains for every $\hat{\mathbb{P}} \in \mathcal{P}_Z \setminus \mathcal{M}_A$, an $X \in C_Z$ such that $\mathbb{E}^{\hat{\mathbb{P}}} X < \inf_{\mathbb{P} \in \mathcal{M}_A} \mathbb{E}^{\mathbb{P}} X = 0$, implying $X \in A$ and $\phi^*(\hat{\mathbb{P}}) = \sup_{X \in C_Z \cap (G-A)} \mathbb{E}^{\hat{\mathbb{P}}} X = +\infty$. So $\hat{\mathcal{M}} \subseteq \mathcal{M}_A$, and as a consequence $\hat{\mathcal{M}} = \mathcal{M}$. This shows that

$$\phi^*(\mathbb{P}) = \sup_{X \in U_Z \cap (G-A)} \mathbb{E}^{\mathbb{P}} X = \begin{cases} 0 & \text{if } \mathbb{P} \in \mathcal{M} \\ +\infty & \text{otherwise,} \end{cases}$$

and it follows from Proposition 2.3 that (2.2) holds. As a result, all conditions (i)–(vi) of Theorem 2.1 are equivalent, which implies that the conditions (i)–(iv) of Proposition 4.2 are equivalent. □

Proof of Proposition 4.4
$\mathrm{Ent}_\lambda^{\mathbb{Q}}$ is a transformed loss risk measure corresponding to the loss function $l_{\mathbb{Q}}(x) = \exp(\lambda x - 1)/\lambda$. Therefore, it follows from condition (4.5) that Lemma 4.1 applies. So we know that condition (2.1) holds. As in the proof of Proposition 4.2, one has

$$\phi^*(\mathbb{P}) \le \sup_{X \in U_Z \cap (G-A)} \mathbb{E}^{\mathbb{P}} X \le \sup_{X \in G,\, \mathbb{E}^{\mathbb{P}} X > -\infty} \mathbb{E}^{\mathbb{P}} X - \inf_{X \in A} \mathbb{E}^{\mathbb{P}} X \quad \text{for all } \mathbb{P} \in \mathcal{P}_Z, \tag{A.10}$$

and $\sup_{X \in G,\, \mathbb{E}^{\mathbb{P}} X > -\infty} \mathbb{E}^{\mathbb{P}} X = 0$ for $\mathbb{P}$ in the set $\mathcal{M}_G$ of all measures in $\mathcal{P}_Z$ satisfying a)–b). Furthermore, since

$$\mathrm{Ent}_\lambda^{\mathbb{Q}}(X) = \sup_{\mathbb{P} \in \mathcal{P}_Z} (\mathbb{E}^{\mathbb{P}}[-X] - \eta_{\mathbb{Q}}(\mathbb{P})) \quad \text{for all } X \in B_Z,$$

where

$$\eta_{\mathbb{Q}}(\mathbb{P}) = \begin{cases} \mathbb{E}^{\mathbb{Q}} \left(\frac{d\mathbb{P}}{d\mathbb{Q}} \log \frac{d\mathbb{P}}{d\mathbb{Q}} \right) / \lambda & \text{if } \mathbb{P} \ll \mathbb{Q} \\ +\infty & \text{otherwise,} \end{cases}$$

one obtains

$$\inf_{\mathbb{Q} \in \mathcal{Q}} \eta_{\mathbb{Q}}(\mathbb{P}) \ge \sup_{X \in B_Z} \left(\mathbb{E}^{\mathbb{P}}[-X] - \sup_{\mathbb{Q} \in \mathcal{Q}} \mathrm{Ent}_\lambda^{\mathbb{Q}}(X) \right) \ge \sup_{X \in A} \mathbb{E}^{\mathbb{P}}[-X] \ge \phi^*(\mathbb{P}) \quad \text{for } \mathbb{P} \in \mathcal{M}_G.$$

It follows from the assumptions that there exists a continuous function $\tilde{\varphi} : [1, +\infty) \to \mathbb{R}$ such that

$$\lim_{x \to +\infty} \frac{\tilde{\varphi}(x)}{x} = +\infty \quad \text{and} \quad \lim_{x \to +\infty} \frac{\varphi(x)}{\tilde{\varphi}(x)} = +\infty.$$

By (4.5), $\mathcal{Q}$ is $\sigma(\mathcal{P}_{\tilde{Z}}, C_{\tilde{Z}})$-compact for $\tilde{Z} = \exp(\tilde{\varphi}(Z))$. Moreover, $\exp(X) \in C_{\tilde{Z}}$ for all $X \in C_Z$, and

$$\mathrm{Ent}_\lambda^{\mathbb{Q}}(X) = \frac{1}{\lambda} \log \mathbb{E}^{\mathbb{Q}} \exp(-\lambda X)$$

is concave and $\sigma(\mathcal{P}_{\tilde{Z}}, C_{\tilde{Z}})$-continuous in $\mathbb{Q}$. Therefore, one obtains from a minimax result, such as e.g. the one of Ky Fan [24], that for all $\mathbb{P} \in \mathcal{P}_Z$,

$$\begin{aligned} \inf_{\mathbb{Q} \in \mathcal{Q}} \eta_{\mathbb{Q}}(\mathbb{P}) &= \inf_{\mathbb{Q} \in \mathcal{Q}} \sup_{X \in C_Z} \Big(\mathbb{E}^{\mathbb{P}}[-X] - \mathrm{Ent}_\lambda^{\mathbb{Q}}(X) \Big) \\ = \sup_{X \in C_Z} &\left(\mathbb{E}^{\mathbb{P}}[-X] - \sup_{\mathbb{Q} \in \mathcal{Q}} \mathrm{Ent}_\lambda^{\mathbb{Q}}(X) \right) = \sup_{X \in C_Z \cap A} \mathbb{E}^{\mathbb{P}}[-X] \le \phi^*(\mathbb{P}). \end{aligned}$$

Since $\phi^*(\mathbb{P}) = +\infty$ for $\mathbb{P} \in \mathcal{P}_Z \setminus \mathcal{M}_G$, this shows that

$$\eta(\mathbb{P}) = \left\{ \begin{array}{ll} \inf_{\mathbb{Q} \in \mathcal{Q}} \eta_{\mathbb{Q}}(\mathbb{P}) & \text{if } \mathbb{P} \in \mathcal{M}_G \\ +\infty & \text{otherwise} \end{array} \right\} = \phi^*(\mathbb{P}),$$

which, by (A.10), implies $\phi^*(\mathbb{P}) = \sup_{X \in U_Z \cap (G - A)} \mathbb{E}^{\mathbb{P}} X$. So Proposition 2.3 gives us that condition (2.2) holds. Now Proposition 4.4 is a consequence of Theorem 2.1. □

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