

# Stressed correlations and volatilities How to fulfill requirements of the Basel Committee

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## Abstract

We propose a new approach to define stress scenarios for volatilities and correlations which fulfills the requirements of the Basel Committee on Banking Supervision for quantifying market risk. Correlations and volatilities are functions of one and the same market factor, which is the key to stress them in a consistent and intuitive way. Our approach is based on a new asset price model where correlations and volatilities depend on the current state of the market. The state of the market captures market-wide movements in equity-prices and thereby fulfills minimum requirements for risk factors stated by the Basel Committee. A maximum likelihood approach is developed to estimate the parameters of the model from market data. For sample portfolios we compare correlations and volatilities in a normal market and under stress and explore consequences on value-at-risk. Stressed value-at-risk exceeds the standard value-at-risk by a factor of 3 to 5, confirming estimates from the Basel Committee.

We finally compare our modeling approach with multivariate GARCH models. For all data analyzed our model turned out to be superior in capturing the dynamics of volatilities and correlations.

**JEL classification codes:** C13, C32, C58, G11, G12

## 1 Introduction

If a bank uses internal models to determine capital requirements for the market risk of a stock portfolio the Basel Committee on Banking Supervision (BCBS) demands that the “bank must calculate a ‘stressed value-at-risk’ measure” where “the relevant market factors were experiencing a period of stress”, see BCBS [2009b] and the updated document BCBS [2011b]. One type of stress tests suggested by the committee

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is to “evaluate the sensitivity of the bank’s market risk exposure to changes in the assumptions about volatilities and correlations”. However, banks that wish to fulfill this requirement face difficult technical questions, which is acknowledged in BCBS [2011a]. One question to be answered is how to model volatilities and correlations of stock returns. In BCBS [2011b] it is explicitly stated that “no particular model is prescribed”. For risk factors the minimum requirement stated in BCBS [2011b] is that “there should be a risk factor that is designed to capture market-wide movements in equity-prices (e.g. a market index)”. However, it is unclear how to use such a factor to stress volatilities and correlations in a consistent manner. Stressing correlations in a portfolio with more than two assets is technically difficult because we need to maintain the positive semi-definiteness of the correlation matrix. In general we cannot increase individual correlations by a fixed quantity without losing the positive semi-definiteness of the correlation matrix. Therefore a naive approach that bases stressed correlations on estimates of bivariate models is infeasible.

Popular models like multivariate GARCH models do not seem to be able to fulfill all the requirements posed by the Basel Committee.

We suggest a model for the dynamics of volatilities and correlations with a market index as the driving risk factor. The model provides a natural setting to define consistent stress scenarios for volatilities and correlations. The stress-scenarios are based on historical experience and correspond to pre-specified probabilities.

In our model the vector of volatilities  $\sigma$  and the correlation matrix  $\rho$  depend on one and the same market state  $F$ . The market state  $F$  is generic; to comply with suggestions of the Basel Committee we define the market state as the realized drift of a market index. We estimate the dependence structure of volatilities  $\sigma(F)$  and correlations  $\rho(F)$  on the market state  $F$  from daily stock prices. In other words, we suggest and estimate a nonlinear one-factor model for the dynamics of volatilities and correlations. Stressed volatilities and correlations are then defined by

$$\sigma(f_\alpha), \quad \rho(f_\alpha),$$

where  $f_\alpha$  is the  $\alpha$ -quantile of the empirical distribution of observed market states  $F$ . The concept is transparent and by construction volatilities and correlations are stressed in a consistent manner. Furthermore, the choice of the quantile probability  $\alpha$  relates to the probability of the stress scenario. We examine consequences of stressed volatilities and correlations on portfolio value-at-risk. We find that the stressed value-at-risk exceeds a standard value-at-risk by a factor of 3 – 5, confirming results in BCBS [2009a].

Our modeling approach is related to existing GARCH approaches. In comparison it turns out that for all given time series that we considered our model captures the behavior of correlations and volatilities better than the Dynamic Conditional Correlation GARCH model by Engle [2002a].

The paper is organized as follows. The model is introduced in Section 2, with estimation technique developed in Section 3. We investigate the dependence structure of volatilities and correlation on the market state  $F$  in Section 4. There we also consider examples for stressed volatilities and correlations and explore consequences for the value-at-risk. The relationship between our model and multivariate GARCH models is discussed in Section 5. In Section 6 we investigate how well different models capture

the dynamics of volatilities and correlations in a given time series. We compare our model with the Dynamic Conditional Correlation GARCH model by Engle [2002b], a model with constant volatilities and correlations, and two moving averages using the past 30 resp. 90 days and find that our model outperforms.

## 2 An asset price model with state-dependent correlation in continuous time

We introduce the setting for an asset price model in continuous time. Our assumption is that asset volatilities and asset correlation depend on the current state of the market, which we may interpret as a common risk factor. An example of the state of the market is the realized drift of a market index, which is determined by past and current asset price realizations. The dependency of the asset dynamics on past asset realizations leads us to consider stochastic differential equations with time delay, so called stochastic delay differential equations (SDDEs), see Mao [2007] or Mohammed [1984]. In every point in time  $t$  volatilities and correlation in our model are determined by states in the past interval  $[t-r, t]$  for some fixed length of the memory window  $r \geq 0$ .

We work on a probability space  $(\Omega, \mathcal{F}, P)$  equipped with a filtration  $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$  satisfying the usual conditions. The filtration is rich enough to carry at least  $n$  independent Wiener processes,  $n \in \mathbb{N}$ . Before we state the dynamics of asset prices  $S = (S^1, \dots, S^n)$  we introduce the concept of the segment process.

**Definition 1.** Let  $S = (S(t))_{t \in [-r, \infty]}$  be a continuous  $\mathbb{R}^n$ -valued stochastic process. For every  $t \geq 0$  we define the  $[t-r, t]$ -segment of the process  $S$  by

$$S_t(u) = S(t+u), \quad u \in [-r, 0],$$

that is,  $S_t$  is a mapping that takes values in the space  $C([-r, 0], \mathbb{R}^n)$  of continuous functions from  $[-r, 0]$  to  $\mathbb{R}^n$ ,

$$S_t : \Omega \mapsto C([-r, 0], \mathbb{R}^n).$$

We call  $(S_t)_{t \in [0, \infty)}$  the segment process with time delay  $r$ . For a segment  $\phi \in C([-r, 0], \mathbb{R}^n)$  we define the norm

$$\|\phi\|_\infty = \sup_{u \in [-r, 0]} \|\phi(u)\|_2,$$

with  $\|\cdot\|_2$  the Euclidean norm on  $\mathbb{R}^n$ .

We base our analysis on the model

$$dS^i(t) = \mu_i(\theta, S_t) S^i(t) dt + \sigma_i(\theta, F(S_t)) S^i(t) dW^i(t), \quad (1)$$

$$dW^i(t) \cdot dW^j(t) = \rho_{i,j}(\theta, F(S_t)) dt, \quad i, j = 1, \dots, n \quad (2)$$

$$S_0 \in C([-r, 0], \mathbb{R}_+^n), \quad (3)$$

with Wiener processes  $W^1, \dots, W^n$ , parameter  $\theta \in \Theta \subseteq \mathbb{R}^p$ ,  $p \in \mathbb{N}$ , drifts and volatility functions

$$\begin{aligned}\mu_i &: \Theta \times C([-r, 0], \mathbb{R}^n) \rightarrow \mathbb{R}, & i = 1, \dots, n, \\ \sigma_i &: \Theta \times \mathbb{R} \rightarrow [0, \infty) & i = 1, \dots, n,\end{aligned}$$

and instantaneous correlation functions

$$\rho_{i,j} : \Theta \times \mathbb{R} \rightarrow [-1, 1], \quad i, j = 1, \dots, n.$$

We assume that the instantaneous correlation matrix

$$\rho(\theta, x) = (\rho_{i,j}(\theta, x))_{i,j=1, \dots, n}$$

is positive definite for all  $(\theta, x) \in \Theta \times \mathbb{R}^n$ , hence there exists a unique Cholesky decomposition. For future reference we define the Cholesky decomposition  $C(\theta, x)$  of the instantaneous covariance matrix, that is,

$$C(\theta, x)C(\theta, x)^T = \text{diag}(\sigma_1(\theta, x), \dots, \sigma_n(\theta, x))\rho(\theta, x)\text{diag}(\sigma_1(\theta, x), \dots, \sigma_n(\theta, x)), \quad (\theta, x) \in \Theta \times \mathbb{R}, \quad (4)$$

with  $\text{diag}(x_1, \dots, x_n)$  a matrix in  $\mathbb{R}^{n,n}$  with entries  $x_1, \dots, x_n$  on the diagonal and zeros otherwise. The volatilities  $\sigma_i$  and the correlations  $\rho_{i,j}$  depend on the market state  $F(S_t)$ , which we describe by a market state function  $F$  of the past segment  $S_t$ ,

$$F : C([-r, 0], \mathbb{R}^n) \rightarrow \mathbb{R}.$$

Since the market state depends on a past window of assets  $S^1, \dots, S^n$ , the model is a system of delay equations.

Before we introduce parameterizations for the drifts  $\mu_i$ , volatilities  $\sigma_i$ , correlations  $\rho_{i,j}$  and the functional form of the market state  $F$  we state sufficient conditions for the existence of a unique solution of our model. Furthermore, we show that the instantaneous correlation can be interpreted as the correlation of daily log-returns. For proofs see Becker and Schmidt [2010].

**Proposition 1.** *Assume that for every  $\theta \in \Theta$  and for  $i = 1, \dots, n$*

$$\mu_i \circ \exp(\theta, \cdot) : C([-r, 0], \mathbb{R}^n) \rightarrow \mathbb{R}$$

*is locally Lipschitz-continuous and fulfills the linear growth condition*

$$|\mu_i \circ \exp(\theta, x)|^2 \leq D(1 + \|x\|_\infty^2), \quad x \in C([-r, 0], \mathbb{R}^n), \quad (5)$$

*with a constant  $D > 0$ . The concatenation  $\mu_i \circ \exp$  is defined as*

$$\mu_i \circ \exp(f) = \mu_i(\exp \circ f^1, \dots, \exp \circ f^n), \quad f = (f^1, \dots, f^n) \in C([-r, 0], \mathbb{R}^n). \quad (6)$$

*Furthermore, assume that for every  $\theta \in \Theta$  and  $i, j = 1, \dots, n$*

$$\begin{aligned}\sigma_i(\theta, \cdot) &: (\mathbb{R}, |\cdot|) \rightarrow ([0, \infty), |\cdot|), \\ \rho_{i,j}(\theta, \cdot) &: (\mathbb{R}, |\cdot|) \rightarrow ([-1, 1], |\cdot|), \\ F \circ \exp &: (C([-r, 0], \mathbb{R}^n), \|\cdot\|_\infty) \rightarrow (\mathbb{R}, |\cdot|)\end{aligned}$$

are locally Lipschitz continuous, where the concatenation  $F \circ \exp$  is defined analogously to  $\mu_i \circ \exp$ . Let the volatility functions  $\sigma_i(\theta, \cdot)$  be bounded, and the instantaneous correlation  $\rho(\theta, x)$  be positive definite for all  $(\theta, x)$ .

Then for every  $\theta \in \Theta$  there exist  $\mathbb{F}$ -adapted Wiener processes  $W^1, \dots, W^n$  and an  $\mathbb{F}$ -adapted process  $S = (S^1, \dots, S^n)$  with strictly positive paths that satisfy the system of delay equations (1)-(3). The distribution of  $S$  on  $C([-r, \infty), \mathbb{R}^n)$  is unique. If the drifts  $\mu_i$  are bounded it holds that

$$E \left[ \sup_{t \in [-r, T]} \sum_{i=1}^n (S^i(t))^2 \right] < \infty. \quad (7)$$

From now on we assume that the conditions of Proposition 1 hold and  $S$  denotes the unique positive solution of the system (1)-(3).

The instantaneous correlation  $\rho_{i,j}$  admits a convenient interpretation as the correlation between daily log-returns of assets  $S^i$  and  $S^j$  given the market state  $F(S_t)$ , that is

$$\rho_{i,j}(\theta, F(S_t)) \approx \text{Corr} \left( \log \frac{S^i(t+1 \text{ day})}{S^i(t)}, \log \frac{S^j(t+1 \text{ day})}{S^j(t)} \middle| \mathcal{F}_t \right).$$

The following lemma makes this more precise.

**Lemma 1.** Denote the components of the Cholesky-decomposition  $C(\theta, x)$  introduced in (4) as

$$C(\theta, x) = (c_{i,j}(\theta, x))_{i,j=1,\dots,n}, \quad x \in C([-r, 0], \mathbb{R}^n).$$

Let the mappings

$$x \mapsto \mu_i(\theta, \cdot) \circ \exp(x), \quad i = 1, \dots, n,$$

and

$$x \mapsto c_{i,j}(\theta, \cdot) \circ \exp(x), \quad x \mapsto c_{i,j}^2(\theta, \cdot) \circ \exp(x), \quad i, j = 1, \dots, n$$

be uniformly Lipschitz-continuous, with the concatenation of functions defined like in (6). Furthermore, let the volatilities  $\sigma_{i,j}$  be bounded and condition (5) hold.

Then for every  $t \geq 0$  and the sequence  $\Delta_m = \frac{1}{m}$ ,  $m \in \mathbb{N}$ , it holds  $P$ -almost surely

$$\lim_{m \rightarrow \infty} \text{Corr} \left( \log \frac{S^i(t + \Delta_m)}{S^i(t)}, \log \frac{S^j(t + \Delta_m)}{S^j(t)} \middle| \mathcal{F}_t \right) = \rho_{i,j}(\theta, F(S_t)).$$

The market state function  $F$  is a common risk factor for the dynamics of volatilities and correlations. The minimum requirement for such a risk factor stated in BCBS [2011b] is that it shall ‘‘capture market-wide movements in equity-prices’’. Therefore we define the market state as a realized drift,

$$\begin{aligned} F(S_t) &= F(S(t), S(t - \Delta t), \dots, S(t - n_F \Delta t)) \\ &= \frac{1}{n} \sum_{j=1}^n \frac{1}{n_F \Delta t} \sum_{k=1}^{n_F} \log \left( \frac{S^j(t - (k-1)\Delta t)}{S^j(t - k\Delta t)} \right) \\ &\quad + \frac{1}{2\Delta t} \widehat{\sigma}^2 \left( \log \left( \frac{S^j(t - (k-1)\Delta t)}{S^j(t - k\Delta t)} \right) : k = 1, \dots, n_F \right). \end{aligned} \quad (8)$$

with

$$\widehat{\sigma}^2(x_1, \dots, x_{n_F}) = \frac{1}{n_F - 1} \sum_{j=1}^{n_F} \left( x_j - \frac{1}{n_F} \sum_{k=1}^{n_F} x_k \right)^2. \quad (9)$$

In case of constant volatilities  $\sigma_i \equiv \sigma_i(\theta, F(S_t))$  and constant drifts  $\mu_i \equiv \mu_i(\theta, S_t)$  the proposed function  $F$  is an unbiased estimator of the average drift  $\frac{1}{n} \sum_{i=1}^n \mu_i$  of the assets  $S^i$ . Alternatively, we can define the market state function  $F$  as the realized drift of a stock market index. It is straightforward to prove that the market state (8) fulfills the conditions in Proposition 1.

For the functional form of volatilities and correlations in the continuous time model (1) - (3) we want to achieve a maximum of flexibility. Parameterizations for certain classes of correlation matrices have been proposed for example by Ding and Engle [2001]. However, it is not clear whether these parameterizations are suitable for our problem. Furthermore, defining parameterizations of positive definite correlation matrices is technically difficult, see for example Rebonato and Jackel [1999]. Therefore we do not define separate parameterizations for correlations  $\rho_{i,j}(\theta, \cdot)$  and volatilities  $\sigma_i(\theta, \cdot)$ . Instead, we introduce a parameterization for the Cholesky decomposition

$$C(\theta, x) = (c_{i,j}(\theta, x))_{i,j=1,\dots,n}$$

of the instantaneous covariance matrix in (4) via

$$c_{i,j}(\theta, x) = \begin{cases} h(x, \xi, \theta, i, j), & i > j \\ \alpha + |h(x, \xi, \theta, i, j)|, & i = j \\ 0, & i < j. \end{cases} \quad (10)$$

The functions  $h(\cdot, \xi, \theta, i, j)$  are cubic splines through a common set of equidistant discretization points  $\xi$  for the values of the market state,

$$\xi = (\xi_l)_{l=1,\dots,n_{\text{Cov}}}, \quad n_{\text{Cov}} \in \mathbb{N}, \quad (11)$$

and individual sets  $(\theta_l^{i,j})_{l=1,\dots,n_{\text{Cov}}}$  of values for every entry of the Cholesky decomposition  $c_{i,j}$ . These individual sets are collected in one common vector  $\theta$  that we estimate from real market data. We choose the points  $(\xi_l)_{l=1,\dots,n_{\text{Cov}}}$  such that they, for given asset realizations  $(s(t_i))_{i=1,\dots,n}$ , cover the range of realized market states

$$\left\{ F(s(t_{i-n_F}), \dots, s(t_i)), i \geq 1 + n_F \right\}.$$

The variable  $\alpha$  is a small, positive number that guarantees that the covariance matrix defined via (10) and hence the correlation matrix  $\rho(\theta, \cdot)$  is positive definite for all  $(\theta, x) \in \Theta \times \mathbb{R}$ .

For the drift functions  $\mu_i$  we use either constant values or the weighted form

$$\mu_i(\theta, S_t) = \sum_{j=0}^m \beta_j \log \left( \frac{S^i(t - j\Delta t)}{S^i(t - (j+1)\Delta t)} \right), \quad (12)$$

with  $\sum_{j=0}^m \beta_j = 1$ ,  $m \in \mathbb{N}$ , and  $\beta_j \geq 0$  for all  $j$ ,

### 3 Estimation method

We estimate the true parameter  $\theta_* \in \Theta$  in the continuous time model (1)-(3) from discrete time, that is daily, market observations. We denote the observations by

$$(s(t_i))_{i=1,\dots,N} = (s^1(t_i), \dots, s^n(t_i))_{i=1,\dots,N}$$

and assume that they are equidistant in time, that is,  $t_{i+1} - t_i = \Delta t = \frac{1}{250}$ .

Statistical methods based on discrete observations for delay equations are not yet well-developed. For delay equations with affine drift a first estimation approach is developed in K uchler and S orenson [2009a] and K uchler and S orenson [2009b]. It is not obvious how these results can be generalized to non-affine drifts, that is, to our model<sup>1</sup>. We propose an approximate maximum likelihood estimator that is heuristically motivated and shown to work well in simulation experiments. A proof of the consistency and asymptotic distribution of this estimator is beyond the scope of this paper and subject of future research.

We base our estimator on realizations of the process  $(\log S(t))_t$ . The log-likelihood function is given by

$$\sum_{i=1+n_F}^N \log p_i \left( \log s(t_i) \mid \log s(t_1), \dots, \log s(t_{i-1}), \theta \right), \quad (13)$$

with  $p_i$  the in our model unknown conditional densities

$$P_\theta \left( \log S(t_i) \in dy_i \mid \log S(t_j) = y_j : j = 1, \dots, i-1 \right) = p_i(y_i \mid y_1, \dots, y_{i-1}, \theta) dy_i.$$

We propose to approximate the density  $p_i$  with the density  $\tilde{p}_i$  of a  $n$ -dimensional normal distribution with parameters motivated by the Euler scheme. As in our setup (8) and (12) we let  $\mu_j(\theta, S_{t_{i-1}})$  and  $F(S_{t_{i-1}})$  only depend on  $S(t_{i-1}), S(t_{i-2}), \dots$ . The Euler scheme for stochastic delay equations is analyzed, for example, in K uchler and Platen [2000]. For non-delay equations our approach reduces to a well-known technique to estimate the parameters of a process whose dynamics is given by an ordinary stochastic differential equation, see for example Hurn et al. [2007].

We define  $\tilde{p}_i$  as the density of a  $n$ -dimensional normal distribution with mean

$$\left( \log S^j(t_{i-1}) + \left( \mu_j(\theta, S_{t_{i-1}}) - \frac{1}{2} \sum_{p=1}^n \tilde{\sigma}_{j,p}(\theta, F(S_{t_{i-1}}))^2 \right) \Delta t \right)_{j=1,\dots,n},$$

and covariance matrix

$$\Delta t C(\theta, F(S_{t_{i-1}})) C(\theta, F(S_{t_{i-1}}))^T$$

introduced in (4).

We consider the approximate maximum likelihood estimator

$$\operatorname{argmax}_{\theta \in \Theta} l(\theta), \quad (14)$$

<sup>1</sup>Private communication with Uwe K uchler.

with likelihood function

$$l(\theta) = \sum_{i=1+n_F}^N \log \tilde{p}_i \left( \log s(t_i) \middle| \Delta t, \log s(t_{i-1}), \dots, \log s(t_{i-n_F}), \theta \right). \quad (15)$$

The computational effort of estimator (14) grows at the order of  $n^3$  with  $n$  the number of assets in the portfolio because we need to compute the inverse of the  $n \times n$ -covariance matrix in the density  $\tilde{p}_i$  of the normal distribution. This strong growth of the computational effort is a common problem also for other sophisticated and popular models like multivariate GARCH models and makes their application to large portfolios practically infeasible, see for example BCBS [2011a]. Inspired by methods in Engle et al. [2009] we reduce the computational effort to order  $n^2$ . This is achieved by replacing the likelihood function (15) by the sum over all bivariate likelihood functions  $l_{k_1, k_2}(\theta)$  that are defined for models with two assets  $S^{k_1}, S^{k_2}$ ,

$$l_{k_1, k_2}(\theta) = \sum_{i=1+n_F}^N \log \tilde{p}_i \left( \log d_{k_1, k_2}(t_i) \middle| \Delta t, \log d_{k_1, k_2}(t_{i-1}), \dots, \log d_{k_1, k_2}(t_{i-n_F}), \theta \right), \quad 1 \leq k_1 < k_2 \leq n. \quad (16)$$

Here,  $\tilde{p}_i$  is the density of a two-dimensional normal distribution with parameters suggested by the Euler scheme and  $d_{k_1, k_2}(t) = (s^{k_1}(t), s^{k_2}(t))$ . Thus we obtain the estimator

$$\hat{\theta}_N = \operatorname{argmax}_{\theta \in \Theta} \sum_{k_1=1}^n \sum_{k_2=k_1+1}^n l_{k_1, k_2}(\theta). \quad (17)$$

To reduce the computational effort further would mean that we have to depart from modeling correlations individually, thereby losing too much flexibility.

The approximate maximum likelihood estimator (14) and thus our estimator (17) can be severely biased, because for large Euler discretization steps  $\Delta t$  the density  $\tilde{p}_i$  of a normal distribution may be a poor approximation for density  $p_i$ . For daily observations we justify the reliability of estimator (17) by re-estimating given parameters from Monte Carlo simulations, see Appendix A. The market state (8) depends on the market memory  $n_F$ , which we estimate via

$$\hat{n}_F = \operatorname{argmax}_{n_F \in \{2, \dots, 250\}} \max_{\theta \in \Theta} \sum_{k_1=1}^n \sum_{k_2=k_1+1}^n \tilde{l}_{k_1, k_2}(\theta, n_F), \quad (18)$$

with

$$\tilde{l}_{k_1, k_2}(\theta) = \sum_{i=251}^N \log \tilde{p}_i \left( \log d_{k_1, k_2}(t_i) \middle| \Delta t, \log d_{k_1, k_2}(t_{i-1}), \dots, \log d_{k_1, k_2}(t_{i-n_F}), \theta \right).$$

See Appendix A for a justification of the reliability of estimator (18).

## 4 Stressed volatilities and correlations and their impact on VaR

We estimate the model (1)-(3) for stocks from the S&P 500 using daily data for the period Jan. 1990 – Nov. 2010. To investigate the functional dependence structure of

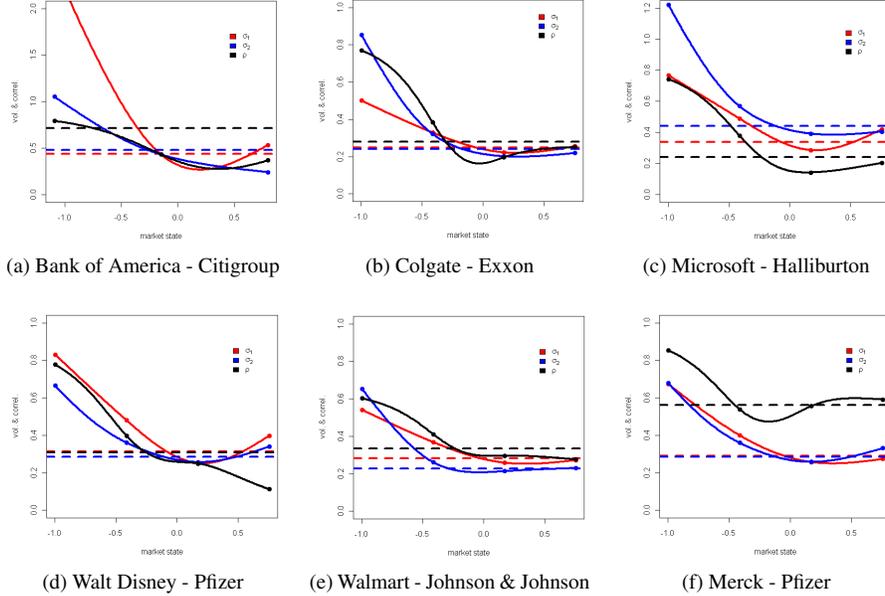


Figure 1: Typical dependency structures of correlation and volatilities on the index-based market state (8) with market memory  $n_F = 75$  business days. Estimates are for US stocks, 1990 – 2010.

asset return volatilities and correlations on the market state we first apply estimator (17) to a model with  $n = 2$  stocks. The estimates in Figure 1 yield that correlations are increased in bear markets and stable in normal and bull markets. Furthermore, correlations and volatilities seem to be co-moving. The dashed lines in Figure 1 are estimated constant correlations and volatilities in a corresponding model that assumes constant volatilities and correlations. The difference between market state dependent volatilities and correlations and their constant counterparts indicates how strongly these quantities change in a crisis. However, we cannot determine stressed correlations for a larger portfolio by computing stressed correlations in a bivariate manner because it is quite likely that we obtain a non-positive-semidefinite matrix. Therefore we estimate the full covariance matrix applying the decomposition (4) and the parameterization (10).

The estimated volatility and correlation structures and the maximum of the likelihood function (17) turn out to be insensitive to the choice of the drift and the weighting scheme (12), which justifies that we use constant drifts  $\mu_i$  in our estimates. For the market memory  $n_F$  we apply estimator (18) and find that the optimal market memory is roughly 75 business days, see Figure 6 in Appendix B.

As a result of our model estimation volatilities and correlations are known (parameterized) functions  $\sigma(F)$ ,  $\rho(F)$  of the market state  $F$ . How can we define stressed volatilities and correlations that comply with the requirements in BCBS [2009b]? We

Vol.	MS	Walmart	GE	Pfizer	Exxon	BoA
	0.517512	0.304813	0.604834	0.345711	0.406842	1.355577
Corr.	MS	Walmart	GE	Pfizer	Exxon	BoA
MS	1.000000	0.391786	0.445221	0.538888	0.620236	0.582697
Walmart	0.391786	1.000000	0.410489	0.529755	0.438121	0.135700
GE	0.445221	0.410489	1.000000	0.588320	0.588359	0.547035
Pfizer	0.538888	0.529755	0.588320	1.000000	0.632803	0.546533
Exxon	0.620236	0.438121	0.588359	0.632803	1.000000	0.406522
BoA	0.582697	0.135700	0.547035	0.546533	0.406522	1.000000

Table 1: Volatilities and correlations for portfolio of six stocks and market state at the 5%-quantile of its empirical distribution. The model is estimated for period Jan. 2004 – Nov. 2010.

propose to define risk scenarios by shifting the market state to predefined quantiles of its empirical distribution. That is, we shift the market state  $F$  to  $\alpha$ -quantiles  $f_\alpha$  with for example  $\alpha \in \{0.1\%, 1\%, 5\%\}$  and compute correlation matrices

$$\rho(f_\alpha) = (\rho_{i,j}(f_\alpha))_{i,j=1,\dots,n} \quad (19)$$

and volatilities

$$\sigma_i(f_\alpha), \quad i = 1, \dots, n. \quad (20)$$

Recall that the market state is defined as the realized drift of an appropriate index, describing the market risk of the stock portfolio. By design the proposed stress scenarios for volatilities and correlations fulfill the minimum requirements posed by BCBS [2009b]. In particular, the choice of the quantile probability  $\alpha$  relates to the probability of the stress scenario. Moreover, volatilities and correlations are stressed in a consistent way, because they are stressed simultaneously and based on one and the same market factor  $F$ .

We illustrate our approach by investigating a portfolio of  $n = 6$  stocks: Bank of America (BoA), Exxon, General Electric (GE), Microsoft (MS), Pfizer and Walmart. Figures 7 and 8 in Appendix C show the estimated dependence structure of volatilities and correlations on the market state. Recall that these estimations must be performed for the portfolio as a whole, and not on a pairwise basis.

Table 1 shows the correlation matrix and volatilities for the market state  $F$  at the 5% - quantile of its empirical distribution, Table 2 for the market state  $F$  at the median. The estimation is based on the period Jan. 2004 – Nov. 2010.

For this example we also consider consequences on the portfolio's value-at-risk. Let us assume that we invest  $V_0 = 100\$$  in our portfolio. For asset weights  $\gamma_i$  we compute a 10-day 'stressed value-at-risk' for level  $\alpha$  by

$$\text{VaR}_\alpha^{\text{stressed}} = -V_0 \left( \exp \left( \sqrt{10} \Phi^{-1}(\alpha) \sqrt{\sum_{i,j=1}^n \gamma_i \gamma_j \sigma_i(f_\alpha) \sigma_j(f_\alpha) \rho_{i,j}(f_\alpha) \Delta t} \right) - 1 \right),$$

where volatilities and correlations are evaluated at the  $\alpha$ -quantile  $f_\alpha$  of the empirical distribution of the market state  $F$ . The function  $\Phi$  is the cumulative distribution

Vol.	MS	Walmart	GE	Pfizer	Exxon	BoA
	0.196424	0.168472	0.200905	0.212423	0.201778	0.285326
Corr	MS	Walmart	GE	Pfizer	Exxon	BoA
MS	1.000000	0.350425	0.437016	0.284452	0.332468	0.238774
Walmart	0.350425	1.000000	0.389480	0.336892	0.287761	0.373636
GE	0.437016	0.389480	1.000000	0.376709	0.384305	0.707337
Pfizer	0.284452	0.336892	0.376709	1.000000	0.299811	0.384892
Exxon	0.332468	0.287761	0.384305	0.299811	1.000000	0.323369
BaA	0.238774	0.373636	0.707337	0.384892	0.323369	1.000000

Table 2: Volatilities and correlations for portfolio of six stocks and market state at the median of its empirical distribution. The model is estimated for period Jan. 2004 – Nov. 2010.

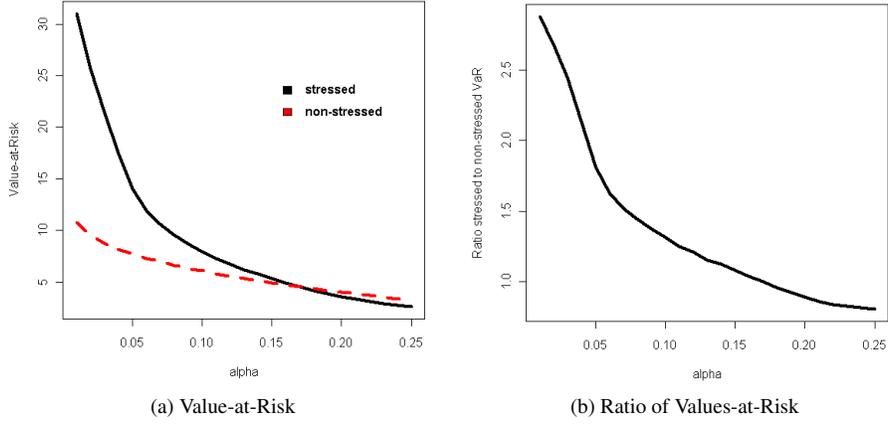


Figure 2: Estimated ten day - VaRs for different probabilities  $\alpha$  for a portfolio of six stocks and portfolio value 100\$. The model is estimated on Jan. 2004 – Nov. 2010.

function of the standard normal distribution. Furthermore, we compute a 'non-stressed value-at-risk' for level  $\alpha$  by

$$\text{VaR}_\alpha^{\text{non-stressed}} = -V_0 \left( \exp \left( \sqrt{10} \Phi^{-1}(\alpha) \sqrt{\sum_{i,j=1}^n \gamma_i \gamma_j \sigma_{i,\text{const}} \sigma_{j,\text{const}} \rho_{i,j,\text{const}} \Delta t} \right) - 1 \right),$$

where volatilities and correlations are estimated from a corresponding model with constant volatilities and correlations. In Figure 2a we plot the functions

$$\alpha \mapsto \text{VaR}_\alpha^{\text{stressed}}, \quad \alpha \mapsto \text{VaR}_\alpha^{\text{non-stressed}}, \quad \alpha \in (0, 0.25) \quad (21)$$

for our portfolio assuming equal asset weights  $\gamma_i$ .

Furthermore, Figure 2b shows the ratio of stressed and non-stressed value-at-risk,

$$\alpha \mapsto \frac{\text{VaR}_\alpha^{\text{stressed}}}{\text{VaR}_\alpha^{\text{non-stressed}}}, \quad \alpha \in (0, 0.25).$$

We observe that the stressed 99%, 10-day value-at-risk  $\text{VaR}_{0.99}^{\text{stressed}}$  computed with market state dependent volatilities and correlations is 3 times higher than the value-at-risk  $\text{VaR}_{0.99}^{\text{non-stressed}}$  computed with constant volatilities and correlations. For a portfolio of 15 stocks<sup>2</sup> we even observe a ratio of 5. Our results reassert the findings of BCBS [2009a], who report that the ratio of the stressed value-at-risk and the non-stressed value-at-risk as computed by banks is in the range of 0.68 – 7 with median 2.6.

We summarize the advantages of our approach. Firstly, the stress scenarios (19)-(20) fulfill the minimum requirements posed by BCBS [2009b]. Moreover, volatilities and correlations are stressed in a consistent way because they are stressed simultaneously and based on one and the same market factor  $F$ . Secondly, the correlation matrix (19) and volatilities (20) can be used as inputs for a market risk analysis in any model where daily returns are assumed to be normally distributed, see the discussion in Section 5. Moreover, as will be shown in Section 6, the model seems to perform better than the Dynamic Conditional Correlation GARCH model by Engle [2002a] in capturing the dynamics of correlations and volatilities within given samples.

## 5 Relation to multivariate GARCH models

Multivariate GARCH models are standard models in the literature to describe the dynamics of volatilities and correlations, see for example Silvennoinen and Teräsvirta [2009]. To compare multivariate GARCH models with our modeling approach we translate the continuous time dynamics (1)-(3) into discrete time. We use the setting introduced in Section 2.

GARCH models assume a discrete time dynamics for stock returns. We introduce the vector of log-returns

$$r(t) = (r_1(t), \dots, r_n(t)),$$

defined as

$$r_i(t) = \log \left( \frac{S^i(t)}{S^i(t-\Delta t)} \right), \quad i = 1, \dots, n,$$

with time  $t \in \{k\Delta t : k \in \mathbb{N}\}$ . Multivariate GARCH models describe the dynamics of  $r$  via

$$r(t) = H^{\frac{1}{2}}(t)\eta(t), \tag{22}$$

with  $H^{\frac{1}{2}}(t)$  the Cholesky decomposition of the  $\mathcal{F}_{t-1}$ -measurable covariance matrix  $H(t) \in \mathbb{R}^{n,n}$  and  $\eta(t)$  a  $\mathbb{R}^n$ -valued random variable with independent, identically dis-

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<sup>2</sup>The portfolio consists of Apple, AT&T, Bank of America, Chevron, Citigroup, Exxon, Ford, General Electric, J.P. Morgan, Johnson & Johnson, Microsoft, Pfizer, Procter & Gamble, Walmart, Walt Disney.

tributed realizations. The variable  $\eta(t)$  has components  $(\eta_1(t), \dots, \eta_n(t))$  with covariances

$$\text{Cov}(\eta_i(t), \eta_j(t)) = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}.$$

We assume that returns  $r_i$  are centered, therefore for every time  $t$  it holds that

$$E(\eta_i(t)) = 0, \quad i = 1, \dots, n.$$

Standard assumptions for the distribution of  $\eta(t)$  are standard normal or t-distributions. For normally distributed shocks  $\eta(t)$  the dynamics (22) implies that

$$r(t) \sim \mathcal{N}(0, H(t)). \quad (23)$$

Multivariate GARCH models state updating rules for the covariance matrix  $H(t)$  based on past observations of the covariance matrix  $H$  and past returns  $r$ . One group of models defines updates for  $H$  that linearly depend on past realizations of covariance matrices  $H$  and past returns  $r$ , for example the VEC-model by Bollerslev et al. [1988]. A second group states the dynamics of the covariance matrix  $H$  indirectly by separately defining the dynamics of volatilities and correlations. In these models the updates of the covariance matrix  $H$  depend on past realizations of  $H$  and  $r$  in a nonlinear way, see for example the Dynamic Conditional Correlation GARCH model by Engle [2002b].

What does the continuous-time dynamics (1)-(3) imply for the discrete time dynamics of returns in equation (22)? For the process  $(\log S(t))_{t \geq 0}$  we define the transition density  $p_i$  for time  $t_i$  given all information up to time  $t_{i-1}$  by

$$p_i(y_i | y_1, \dots, y_{i-1}, \theta) dy_i = P_\theta \left( \log S(t_i) \in dy_i \mid \log S(t_j) = y_j : j = 1, \dots, i-1 \right).$$

The transition density  $p_i$  is unknown in our model. Like in Section 3 we approximate  $p_i$  with the density of an  $n$ -dimensional normal distribution with distribution parameters implied by the Euler scheme. For the distribution of asset returns in time  $t_i$  given the information up to time  $t_{i-1}$  we obtain

$$r(i\Delta t) \approx \mathcal{N}(\mu((i-1)\Delta t), H((i-1)\Delta t)) \quad (24)$$

with

$$\begin{aligned} \mu((i-1)\Delta t) &= \left( \mu_k(\theta, S_{(i-1)\Delta t}) - \frac{1}{2} \sigma_k^2(\theta, F(S_{(i-1)\Delta t})) \right)_{k=1, \dots, n} \Delta t \\ H((i-1)\Delta t) &= \Delta t \sigma(\theta, F(S_{(i-1)\Delta t})) \rho(\theta, F(S_{(i-1)\Delta t})) \sigma(\theta, F(S_{(i-1)\Delta t})) \end{aligned} \quad (25)$$

and

$$F(S_{(i-1)\Delta t}) = F(S((i-1)\Delta t), \dots, S(i-1-n_F\Delta t)).$$

We neglect the drift  $\mu((i-1)\Delta t)$  in (24) and obtain a discrete time dynamics like in (23), that is,

$$r(t) \approx \sqrt{\Delta t} C(t - \Delta t) \eta(t), \quad (26)$$

with  $C(t - \Delta t)$  the Cholesky decomposition of the instantaneous covariance introduced in (4), and  $(\eta(t))_t$  independent identically distributed random vectors with  $n$ -dimensional standard normal distribution. Hence our model implies a discrete time dynamics whose structure is analogous to (22). We find two similarities between the dynamics suggested by (24)-(25) and multivariate GARCH models. First, the covariance matrix (25) is decomposed in volatilities and correlations like in a DCC-GARCH model. Second, the market state function  $F$  as defined in (8) can be expressed not only as a function of price realizations  $S((i-1)\Delta t), \dots, S((i-1-n_F)\Delta t)$  but also as a function of realized returns  $r((i-1)\Delta t), \dots, r((i-n_F)\Delta t)$ . Hence like GARCH models the model dynamics suggested by (24)-(25) operates on returns instead of price observations. Furthermore, like in a DCC-GARCH model volatilities and correlations are non-linear functions of realized returns. More precisely, volatilities and correlations are non-linear functions of a factor  $F$  that depends on finitely many past returns  $r((i-1)\Delta t), \dots, r((i-n_F)\Delta t)$ .

## 6 Capturing the dynamics of volatilities and correlations: a comparison with other models

How well does the model (1)-(3) describe the dynamics of volatilities and correlations? To answer this question we follow an approach by Engle and Colacito [2006] and Engle [2008] who use different models to optimize an asset portfolio based on models' predicted volatilities and correlations; the model that yields the smallest portfolio variance is best able to capture the dynamics of volatilities and correlations.

At each time step  $t - \Delta t$  we re-allocate assets based on models' predictions of the covariance matrix  $H(t) = (h_{i,j}(t))$ . We use the notation introduced in Section 5. We denote the asset weights by

$$w(t) = (w_1(t), \dots, w_n(t)) \in \mathbb{R}^n,$$

and the return of the asset portfolio is

$$r_{\text{portfolio}}(t) = \sum_{i=1}^n w_i(t) r_i(t).$$

If the sum  $\sum_{i=1}^n w_i(t)$  differs from one we invest the difference

$$1 - \sum_{i=1}^n w_i(t)$$

in a riskless asset. We neglect this investment as it does not affect the variance of our portfolio. For every time  $t$  our investment problem is

$$\begin{aligned} \min_{w(t) \in \mathbb{R}^n,} \quad & w^T(t) H(t) w(t), \\ \text{s.t.} \quad & w^T(t) \mu = \mu_0 \end{aligned} \tag{27}$$

where  $w^T(t)$  denotes the transpose of vector  $w(t)$ ,  $\mu \in \mathbb{R}^n$  is a vector of by assumption constant expected returns, and  $\mu_0 \in \mathbb{R}$  is the required return of our portfolio. The

	DCC	Const	Avg30	Avg90	ContinTime
MinVar	0.220573	0.223925	0.231529	0.222258	0.219494
Hedge	0.293514	0.293704	0.309974	0.297335	0.291792

Table 3: Average portfolio volatilities, 1997 – 2004.

	DCC	Const	Avg30	Avg90	ContinTime
MinVar	0.174216	0.172720	0.177059	0.172185	0.165524
Hedge	0.183751	0.182105	0.192098	0.183896	0.178851

Table 4: Average portfolio volatilities, 2004 – 2010.

solution to problem (27) is

$$w(t) = \frac{H^{-1}(t)\mu}{\mu^T H^{-1}(t)\mu} \mu_0.$$

We analyze two portfolio strategies. The first strategy is to set up a minimum variance portfolio. Following the argument of Engle and Colacito [2006] this means that we assume that the vector of expected returns is

$$\mu = (1, \dots, 1) \in \mathbb{R}^n.$$

In the second portfolio strategy we follow a buy-and-hold strategy in the first asset while all other assets are used to hedge the first one. The vector of expected returns that corresponds to this strategy is

$$\mu = (1, 0, \dots, 0) \in \mathbb{R}^n.$$

In the following we consider portfolios that consist of four stocks. We compare five different models: the discrete time approximation (26) of our continuous time model (in the following tables denoted by 'ContinTime'); the standard mean-reverting Dynamic Conditional Correlation GARCH model (denoted by 'DCC') by Engle [2002a] and according to Engle [2008] one of the best performing models for anticipating correlations; a model with constant volatilities and correlations ('Const'); and two moving averages over the past 30 resp. 90 days ('Avg30' resp. 'Avg90'). We estimate the models on time periods Jan. 1997 – Jan. 2004 and Jan. 2004 – Nov. 2010 for all sets of four stocks out of ten large cap US-stocks<sup>3</sup>. We want to investigate how well the models capture the dynamics of volatilities and correlations in a given period. Therefore the portfolio volatilities are computed on the same time periods as the models are estimated on. Tables 3 and 4 show average realized portfolio volatilities over all stock portfolios.

The differences in portfolio volatilities are small but systematic. Tables 5 and 6 show the percentage of asset pairs for which the model in the respective row yields a smaller portfolio volatility than the model in the respective column. We observe that the model (26) has a clear and systematic advantage over all other models.

<sup>3</sup>We use AT&T, Coca Cola, Exxon, Ford, General Electric, Johnson & Johnson, Microsoft, J.P. Morgan, Procter & Gamble, Walmart.

MinVar	DCC	Const	Avg30	Avg90	ContinTime
DCC	–	0.847619	1.000000	0.795238	0.314286
Const	0.152381	–	0.952381	0.361905	0.000000
Avg30	0.000000	0.047619	–	0.000000	0.014286
Avg90	0.204762	0.638095	1.000000	–	0.204762
ContinTime	0.685714	1.000000	0.985714	0.795238	–

Hedge	DCC	Const	Avg30	Avg90	ContinTime
DCC	–	0.571429	1.000000	0.985714	0.223810
Const	0.428571	–	1.000000	0.895238	0.014286
Avg30	0.000000	0.000000	–	0.000000	0.000000
Avg90	0.014286	0.104762	1.000000	–	0.019048
ContinTime	0.776190	0.985714	1.000000	0.980952	–

Table 5: Percentage of asset portfolios for which the model in the row yields a smaller portfolio volatility than the model in the column. Estimates are for period 1997 – 2004.

MinVar	DCC	Const	Avg30	Avg90	ContinTime
DCC	–	0.319048	0.666667	0.352381	0.052381
Const	0.680952	–	0.747619	0.490476	0.000000
Avg30	0.333333	0.252381	–	0.133333	0.000000
Avg90	0.647619	0.509524	0.866667	–	0.028571
ContinTime	0.947619	1.000000	1.000000	0.971429	–

Hedge	DCC	Const	Avg30	Avg90	ContinTime
DCC	–	0.190476	0.985714	0.514286	0.009524
Const	0.809524	–	1.000000	0.780952	0.000000
Avg30	0.014286	0.000000	–	0.000000	0.000000
Avg90	0.485714	0.219048	1.000000	–	0.000000
ContinTime	0.990476	1.000000	1.000000	1.000000	–

Table 6: Percentage of asset portfolios for which the model in the row yields a smaller portfolio volatility than the model in the column. Estimates are for period 2004 – 2010.

## A Reliability of the Euler-based maximum likelihood estimator

We test the reliability of estimators (17) and (18) by re-estimating a given parameterization in model (1) - (3) from simulated discrete time asset realizations. We use  $n = 4$  assets and  $n_{\text{Cov}} = 8$  discretization points for every spline function of the instantaneous covariance matrix. In the model (1) - (3) we use market state function (8) with  $n_F = 75$  days and constant drifts  $\mu_i = 0.1$  for all  $i = 1, \dots, 4$ . For the dependency of volatilities and instantaneous correlation on the market state we define a Cholesky decomposition  $H(F(S_t)) = (h_{i,j}(F(S_t)))_{i,j}$  of the instantaneous covariance matrix via

$$h_{i,j}(x) = \begin{cases} \frac{2}{\pi} \arctan(\sin(i+j+0.9x)), & i > j \\ 0.001 + \left| \frac{2}{\pi} \arctan(\sqrt{|2j+0.9x|}) \right|, & i = j \\ 0, & i < j, \end{cases} \quad i, j = 1, \dots, n.$$

We use an Euler scheme to generate a time series of daily asset realizations with  $\Delta t = \frac{1}{250}$  over a period of 20 years. To keep discretization bias small in the Monte Carlo simulation we use 2000 additional equidistant discretization steps per day. Figures 3

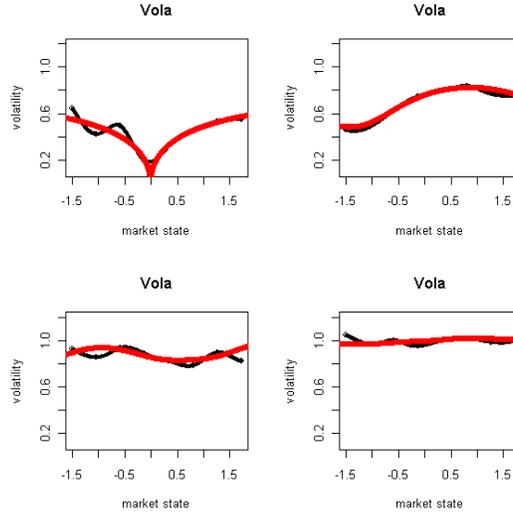


Figure 3: Re-estimation of pre-specified dependency structures of volatilities on the market state. Black lines indicate the empirical estimate (17), red lines indicate the true model dependencies.

and 4 show that estimator (17) yields a reliable estimate of the dependency structure of volatilities and correlation on the market state. Figure 5 shows that estimator (18) is able to identify the market memory.

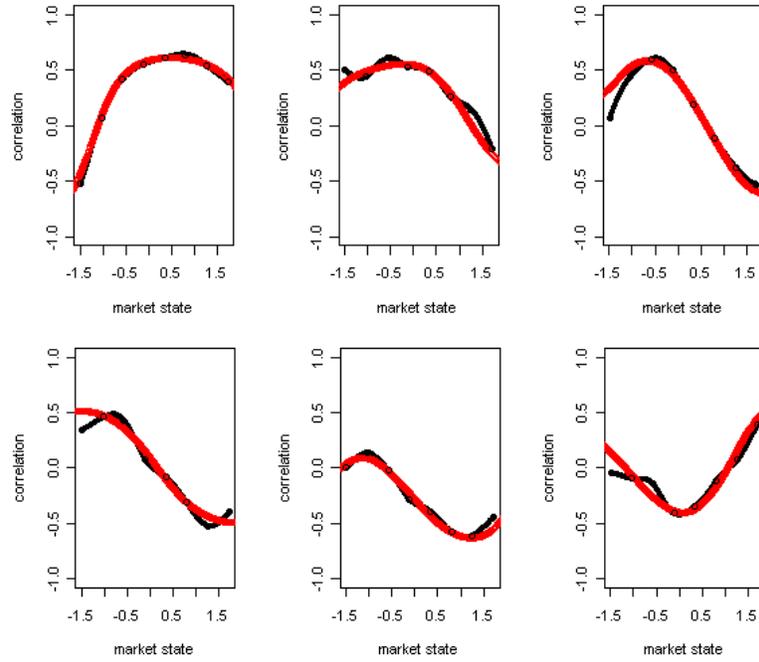


Figure 4: Re-estimation of pre-specified dependency structures of correlation on the market state. Black lines indicate the empirical estimate (17), red lines indicate the true model dependencies.

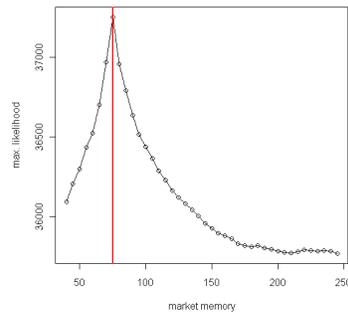


Figure 5: Re-estimation of the market memory  $n_F$  of the market state function  $F$  in the model (1) - (3). The Black line indicates the empirical estimate (17), the red line indicates the true market memory.

## B Empirical estimates of the market memory

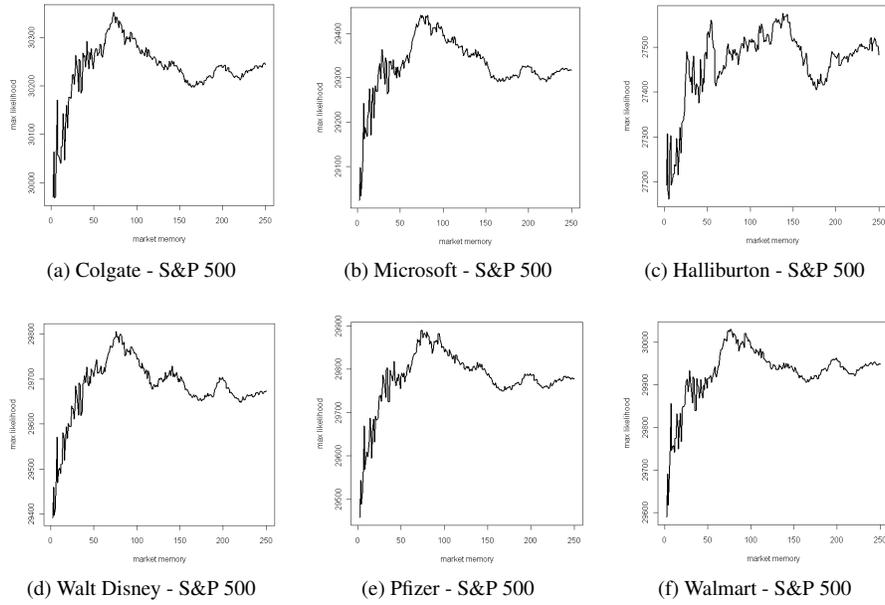


Figure 6: Estimates for US data, 1990 – 2010, yield that the estimated market memory  $\bar{n}_F$  is approximately 75 business days.

## C Empirical estimates of volatilities and correlations

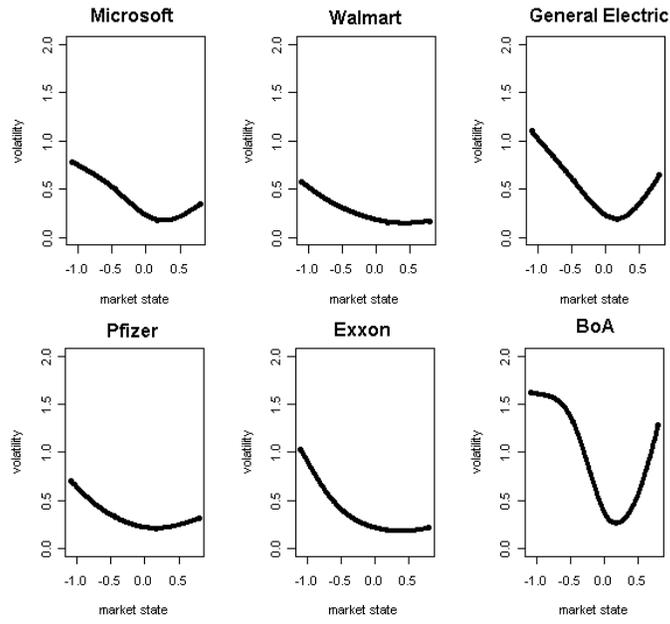


Figure 7: Dependence structure of volatilities on the market state for a portfolio of six stocks. Estimates are for Jan. 2004 – Nov. 2010.

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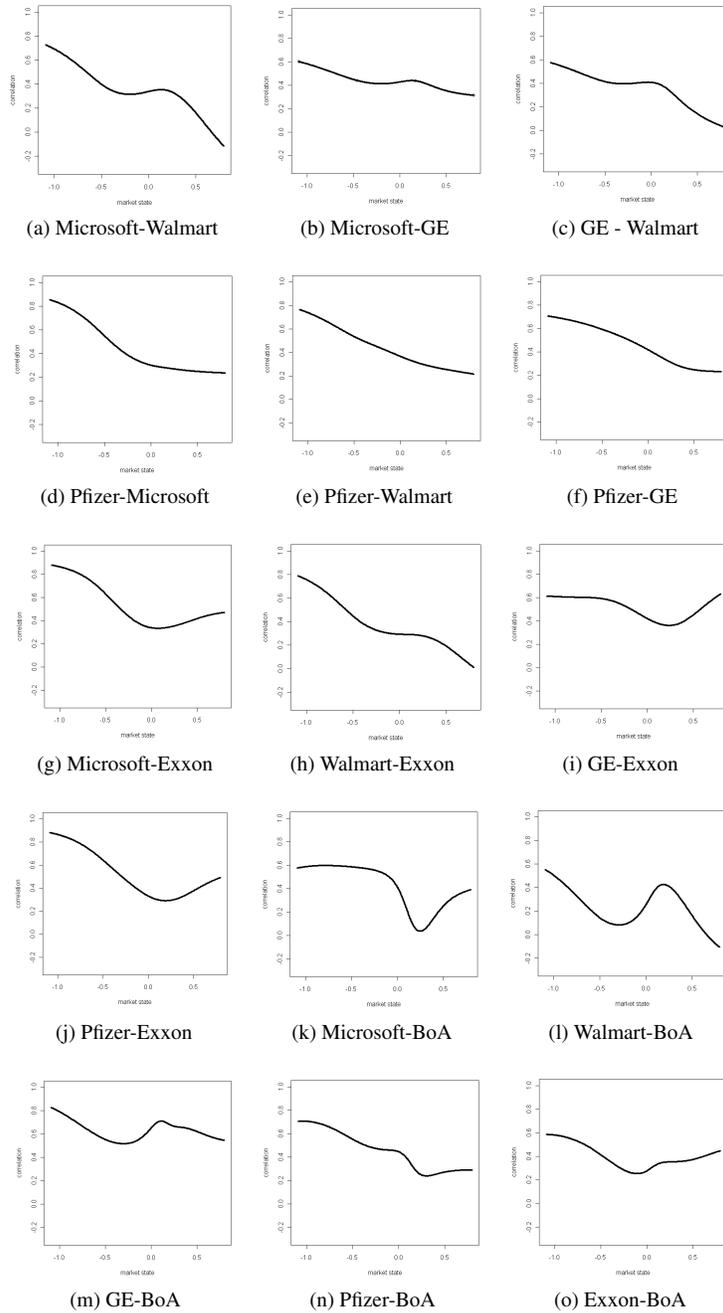


Figure 8: Dependence structure of correlations on the market state for a portfolio of six stocks. Estimates are for Jan. 2004 – Nov. 2010.